WHEN PRISONERS ENTER BATTLE:  
NATURAL CONNECTIONS IN 2 X 2 SYMMETRIC GAMES

by:

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Recently, game theory has gained much interest from many mathematicians as well as economists and psychologists. The simplest and most popular games studied in this field are the 2 X 2 games, which involve two players with two different choices each; each player makes his or her decision individually, but his or her choice will affect the outcome for both that player as well as for the other player. Within these 2 X 2 games, each player has his or her own preference in terms of what outcomes are best for them; for each of the four possible outcomes in these 2 X 2 games, each player also receives a certain payoff, which could be a good or bad payoff. If both players have the same ordering of outcomes, then the game is called symmetric, meaning if the players were switched, the outcomes would be in the same order as initially.

In this paper, I show the results I found while researching the connections between these symmetric 2 X 2 games. The twelve total symmetric 2 X 2 games can be shown on a 2D x-y axis; these games can be separated into six different sectors. In each section, the games involved can be manipulated, when transitioning to another game with different payoff preferences, to one common game. When one game is changed to another by simply swapping two of the payoffs, a transition game in between these games appears; by doing a simple operation to these transition games, I was able to find one universal game in each sector. This proves that these are more closely related than mathematicians previously believed.

If one has an interest in game theory, wants to learn about an interesting topic in mathematics, or just wants to see what one can do with the power of mathematics, one can read all about the 2 X 2 symmetric games in *When Prisoners Enter Battle: Natural Connections in 2 X 2 Symmetric Games*. 
I: Introduction

“In terms of the game theory, we might say the universe is so constituted as to maximize play. The best games are not those in which all goes smoothly and steadily toward a certain conclusion, but those in which the outcome is always in doubt.” This quote by George B. Leonard provides a telling indication of the interest that game theory has attracted. Game theory analyzes situations in which decision-makers interact. It is used to clarify economic, political, and biological occurrences. Game theory has been used to examine situations like the amount of work partners should do individually in a group project, the ideas of companies to produce either high definition or blue ray movies, and even the decision of a country to make atomic bombs. “A game of strategy is a model of a situation involving conflicts of interests” (Rapoport 1). Game theory studies these games. Invented by John von Neumann and Oskar Morgenstern, the field is meant for people who want to explore interactions between multiple players making decisions that can affect the other player. Game theory is applicable to any situation in which one can use mathematics to describe actions performed by players whose behaviors affect the outcomes of others. One game can involve any number of players, and each of those players has any number of possible actions from which to choose. Each decision-maker also assigns a specific preference order to his or her possible outcomes to take.

A game can sometimes be represented by a payoff matrix, which shows each player’s possible actions and the payoff for each outcome depending on all the players’ choices. A payoff matrix is an interpretation of a strategic game. By studying the payoff matrix of a game, one can understand the dynamics of the game. In a game with two players who each have two different options to choose from, a 2 X 2 game, there are four different possible outcomes, each with a different payoff for the players, so each outcome has two numbers: the outcome payoff for the first player and the second player.
One of the most popular games known in game theory is Prisoner’s Dilemma. In this game, it is understood that two criminals have been captured and will be questioned separately to determine who is responsible both for a small misdemeanor and a felony. If neither of them confesses, or stays quiet, nor betrays, or finks, the other prisoner by incriminating him or her, then they will both be convicted for the minor crime and spend the minimum time in jail. This outcome would be the best for both players collectively. However, if one prisoner finks on the other, then that prisoner will be released, and the other will be convicted and spend the maximum of time in prison. The prisoner who finks would get the best possible outcome for him or herself only, while the other prisoner would get the worst. If both fink and state that the other was involved, then they will both spend a long, though not the longest, period of time in prison.

Based on this information, each player has his or her own preference order. In Prisoner’s Dilemma, each player would prefer most for him or her to fink on the other while the other stays quiet. The players’ second choice would be for both prisoners to stay quiet; that way, they would only spend about a year in jail. Next, each prisoner would prefer for both players then to fink on each other. This means time in jail for both, but less time than when only one finks on the other. Lastly, each player’s worst preference is for the other to fink on him or her while he or she stays quiet. So the payoff matrix for this game shows both player’s choices and the outcomes for each player. The possible payoffs are 0, 1, 2, and 3, with 0 being the worst and 3 being the best.
The preference orders are listed for the first player first, then the second player. In a preference, \((x, y)\), \(x\) represents the payoff for player one in this game if that particular situation were to occur, and \(y\) represents player two’s payoff. For example, if player one were to stay quiet and say nothing about the crimes while the second player finks on player one, then player one’s payoff would be 0, meaning that he or she would receive the worst possible outcome for him or her; player two would receive a payoff of 3, meaning that he or she would collect the best possible outcome for him or her. This game represents just one of many games in game theory.

In the 1940’s and 1950’s, John Nash completed work that transformed the field of game theory forever. For one thing, he established the notion of a Nash Equilibrium, which is an outcome in which neither player can do better by changing his or her decision in a game if every other player remains at his or her current choice. In this situation, neither player has any incentive to change his or her choice; changing would only cause a worse outcome for the player contemplating changing his or her choice. In Prisoner’s Dilemma, the one Nash Equilibrium is when both players fink, when the outcome happens to be \((1, 1)\). If player one stays at fink, then player two could either stay at fink or switch to quiet; if he or she moves to quiet, then his or her outcome would change to 0, which is worse than 1. So player two would benefit from just staying where he or she is. The same explanation applies to the situation where player two...
remains at the decision to fink while player one has to decide whether to switch his or her choice or not.

Games are considered to be symmetric if players can change identities without the payoffs changing as well. If the players switched places, then their preferences would still be in the same order as originally. “If both players have strategic choices X and Y, payoffs for the symmetric games are restricted in a simple but very strong way: \( r(x, y) = c(y, x) \)” (Robinson 58). Games that are symmetric have the form:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>((k, k))</td>
<td>((l, m))</td>
</tr>
<tr>
<td>Y</td>
<td>((m, l))</td>
<td>((n, n))</td>
</tr>
</tbody>
</table>

If one game can be changed to another game by just switching one of the preferences, then those games are considered close to one another. For example, starting with a symmetric game as shown above or in Prisoner’s Dilemma, one could switch player one’s \(k\) and \(l\) payoffs to create a neighbor of the original game. Games that are connected to each other by just one preference are considered to be neighbors of the original game. If player one’s \(k\) and \(l\) payoffs were switched, the new game is a neighbor of the initial game. All the games of a neighborhood, which include a game and its neighbors, can be considered family games. For example, a family of games exists with a table, where X and Y are the actions from which the players can choose, with these ordinal preferences:
For Prisoner’s Dilemma, the table is the one given when $a = 0$ and $b = 0$, and in a game known as Chicken, the table of ordinal preferences is given when $a = 1$ and $b = 0$. Chicken and Prisoner’s Dilemma are neighbors; when plugging in the appropriate values for $a$ and $b$, the game for Prisoner’s Dilemma is described by $2, 2; 0, 3/3, 0; 1, 1$ (as shown in the first table), and Chicken is described by $2, 2; 1, 3/3, 1; 0, 0$. This game occurs when, starting at Prisoner’s Dilemma, one simply switches the 0 and 1 preferences. As described before, Prisoner’s Dilemma has one Nash Equilibrium at $(1, 1)$; Chicken, however, while a neighbor of Prisoner’s Dilemma, has two Nash Equilibria at $(1, 3)$ and $(3, 1)$. This shows that by going from one game to another, differences arise. But besides this, one can see that both Prisoner’s Dilemma and Chicken are symmetric; they both fit the correct form to be considered symmetric.

One can also look at the game in between these two. If, say, $a = \frac{1}{2}$ and $b = 0$, the game becomes $2, 2; \frac{1}{2}, 3/3, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}$. By turning this into strictly integers, one can look at it as $\frac{1}{2} < 2 < 3$, so this can be equivalent to the integer inequality $0 < 1 < 2$; therefore, the game between Prisoner’s Dilemma and Chicken can be written as $1, 1; 0, 2/2, 0; 0, 0$. This game has three Nash Equilibria; $(0, 2), (2, 0)$, and $(0, 0)$ are all situations in which both players can do no better just by changing their own choices if the other player does not change. By looking at this, one can possibly see how Prisoner’s Dilemma, a game with just one Nash Equilibrium, can turn into Chicken, a game with two Nash Equilibria. Also, Prisoner’s Dilemma, a game with four
preferences for each player turns into a game with just three preferences, which in turn becomes a game with four preferences again. Games and their neighbors, along with games in between a game, will be examined in more detail in subsequent chapters.

Another way to represent games, besides as a payoff matrix, is a graph with x and y axes having the payoffs as plotted points, double lines connecting row, or player one, points, and single lines connecting column, or player two, choices. Nash Equilibria can be denoted with an open dot instead of a closed dot as the plotted point. Throughout this paper, one can see games using these graphs. For Prisoner’s Dilemma, one can turn the payoff matrix into a graph by connecting the points, (2, 2) with (3, 0) and (0, 3) with (1, 1), using double lines and the points, (2, 2) with (0, 3) and (3, 0) with (1, 1), using single lines. The order graph for Prisoner’s Dilemma looks like:

![Figure 1 - Prisoner’s Dilemma](image)

One can see the path from Prisoner’s Dilemma to Chicken in these graphs:

![Figure 2 - Path from Prisoner’s Dilemma to Chicken](image)

In this paper, I will look at 2 X 2 games that are symmetric in close detail to determine how the games come together and how they are connected. I will analyze how games can change from one to another in preferences. I will also show the order graphs to illustrate Nash
Equilibria. Because some games are so close to each other, I want to look at how they are related in terms of operations and what happens in between the changes from one game to another. By picturing the Nash Equilibria, I can see just how different or similar the games really are. The transition between two games is something that I intend to focus on. Seeing the shift between two games, I will be able to provide a descriptive explanation and depiction of the relations between symmetric 2 X 2 games.
II: Background

“The 2 X 2 games are usually the first that students meet and probably the last they forget” (Robinson 1). A game in game theory includes merely a set of players interacting with each other and making decisions based on a set of rules; as stated in the previous chapter, a 2 X 2 game focuses on only two players with two options from which they can each choose. This means that there are four outcomes per game, and each outcome can be described by a distinct payoff for each participant; the complete game can be explained by just eight numbers. They are the simplest games to examine and analyze in game theory. Because of this, they are also compelling because there are so many types of 2 X 2 games, even though they have a relatively small amount of information included in them. Robinson and Goforth, in their The Topology of the 2 X 2 Games: A New Periodic Table, try to provide a logical and efficient structure for the 2 X 2 games. They explain that relationships between all the 2 X 2 games have never been outlined and that there are many natural connections between all of these simple 2 X 2 games.

When a game is written as a payoff matrix to see the four possible outcomes (eight payoffs, four for each player), it can be written as one of four matrices that has the same payoff pairs, just arranged opposite or diagonally. For example, Prisoner’s Dilemma, presented in the first chapter, has the payoffs (ignoring the actions):

<table>
<thead>
<tr>
<th>Player 1</th>
<th>Player 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 0)</td>
<td>(2, 2)</td>
</tr>
<tr>
<td>(0, 3)</td>
<td>(1, 1)</td>
</tr>
</tbody>
</table>
If one simply switched the payoffs in the columns, once again ignoring the actions, the game would be the same:

```
Player 2
(0, 3) (2, 2)
(1, 1) (3, 0)
```

Player 1

One could also change the rows:

```
Player 2
(3, 0) (1, 1)
(2, 2) (0, 3)
```

Player 1

Finally, one can change payoffs diagonally:

```
Player 2
(0, 3) (2, 2)
(1, 1) (3, 0)
```

Player 1

By interchanging rows, columns, or diagonals, one can see the four possible matrices. Together, the payoff matrices match, but they just have different labels. “The games will be behaviourally equivalent” (7). Robinson and Goforth explain that Anatol Rapoport and Melvin Guyer counted the total number of 2 X 2 games by first figuring out how many possible combinations of two sequences, for the payoffs of two players, of four numbers, for the four possible preferences; this number came out to be 576. Although, of these, many games can be
duplicates just by starting from any of the four possible outcomes. Because of this, they divided 576 by four to get 144.

In this number, twelve games are symmetric, meaning that the game is the same for both players; the preferences for both players are in the same order. The way to count twelve is similar to how Rapoport and Guyer, as well as Robinson and Goforth now, proceeded about counting all the 2 X 2 games. Two cases that fit for the condition stated are possible, \( y = x \) or \( y \neq x \). The first one takes place when the two players choose the same strategy, in which they get the same payoff. For the second case, this situation occurs when the players choose different strategies. One can easily count the number of ways of finding a symmetric game simply by counting the number of ways that two different payoffs can be situated in the game. Since the payoffs are 0, 1, 2, or 3, with 0 being the lowest and 3 being the best possible payoff for each player, there are four different potential payoffs for each player. The number of symmetric 2 X 2 games can be calculated by counting the number of possible combinations of having four items, which are for the four payoffs, and choosing two of them out of the four, for the two strategies where the payoffs have to be different in the game. This number comes out to 12 (59).

Furthermore, this means that there are 144 – 12 = 132 asymmetric games. Additionally, Rapoport and Guyer argued that two games can still be the same if one is a “reflection” of the other; if one simply switches the two players’ preferences, then the games are equal. If players behave in the same way for the same payoff structure, then the players are indistinguishable. All of the asymmetric games have reflections. Taking this into consideration means that there are only 132 divided by 2 asymmetric games, which equals 66 asymmetric games. 66 (the asymmetric games) + 12 (the symmetric games) equals 78 individual games. According to Rapoport and Guyer, there are 78 total 2 X 2 games (16-19).
According to Robinson and Goforth’s representation of all the games, however, they propose not just to look at the 78 games with the reflections being equivalent, but to look at all 144 games. They do not consider the players to be interchangeable; each player should have his or her own preference order for each game. With this, they are able to come up with a system which links them; “Every game is related to every other in the sense that there is a transformation that converts the payoff structure for one into the payoff structure for the other” (33). They explain that if the players switch their preferences with each other, then that is indeed a new game, unlike in the Rapoport and Guyer system.

Each player has three possible switches; one can change a player’s 0 and 1 payoffs, 1 and 2 payoffs, or 2 and 3 payoffs. Since both players can do three different switches, each game has six (two players times three swaps) closest neighbors that are different from the original game in only a small ordering of the outcomes. A row or a column preference can be swapped to create a new game that is reached only by a minimal variation in one player’s data. These six games that result from one single swap of two consecutive preferences make up a game’s neighborhood. With this information, one can form a graph with games as the vertices and the six swaps of a game as the six edges that meet at each vertex. If one removes a swap from the graph, then one removes edges. By breaking up this graph into many different groupings, one can see above layers, slices, pipes, and hotspots.

A layer contains 36 ordinal games that are reached by the operations $R_{01}$ (switching player one’s, also called the row player, 0 and 1 preferences), $C_{01}$ (switching player two’s, also called the column player, 0 and 1 preferences), $R_{12}$, and $C_{12}$. There are four layers (36 times 4 equals 144), and all games in a layer have the same layer index, which is the first number in the ordinal 2 X 2 game, depending on the first player’s preference for the upper left (in a payoff
matrix as shown for Prisoner’s Dilemma) possible outcome: 0, 1, 2, or 3. On Layer 1 in Robinson and Goforth’s representation, Prisoner’s Dilemma and Chicken are present. Layer 2 includes games in which the column player, or player two, normally does better, while Layer 4 shows games where the row player, or player one, does well. Layer 3 is comprised of the games in which the best outcome is realistic for both players. The $R_{23}$ (switching player one’s 2 and 3 preferences) and $C_{23}$ (switching player two’s 2 and 3 preferences) swaps always lead to games on another layer; this is because the operations that create a layer always leave the highest preference, 3, untouched (51).

If the operation, $C_{23}$, is combined with $C_{01}$ or $R_{23}$ with $R_{01}$, in the graph, one will shift to a game exactly above or below the original game on another layer. This new game has in common the same row and column indices with the starting game; these games create a stack. Slices consist of games with a fixed payoff pattern for one player paired with all possible payoff patterns for the opponent. If one player knows his or her payoff structure, then that means that any of 24 possible games exist and could happen and, therefore, appear in the slice (48–49).

The four layers can be observed collectively as four nested toruses connected by the $C_{23}$ and $R_{23}$ operations. In these layers, if one were to start with an initial game and then swap player one’s 0 and 1 payoffs, one would see a different game. Similarly, if one swaps player two’s 0 and 1 payoffs again from the original game, that comprises another game. Finally, if one switched both player one and player two’s 0 and 1 preferences together, that is a fourth different game. These four games make up a tile, four games that differ only by 0 and 1 swaps. Tiles are usually composed of games that are similar in the players’ behaviors. When swapping any player’s 1 and 2 payoffs, then one moves to another tile. Along with this, when $R_{23}$ and $C_{23}$
switches connect to the same layer, hotspots arise. Likewise, pipes occur with four tiles, and they happen when the \( R_{23} \) and \( C_{23} \) link to different layers (54-55).

Within the layers, games have “patterns of conflict and common interest” (119). Some games are pure conflict games, games in which a change for one player that betters his or her payoff will lessen the other player’s payoff. Other games are pure common-interest, which means that if one player changes his or her strategy to better the payoff received, then the other player will also collect an improved payoff. Lastly, there are “Type” games, where the same game can be considered a game of pure common-interest for one player, but a game of pure conflict for the other player. “In Type games, the players are asymmetric in the strongest way. One player always gains by making the other better off; the other always loses by making her partner better off” (123-124). Type games have never been categorized as a distinct class. Using this information, Robinson and Goforth present a “complete classification of the ordinal 2 X 2 games in terms of the degree of conflict among the players” (129) and argue that Rapoport and Guyer do not properly use conflict in order to connect all the 2 X 2 games. Because of conflict, one can see many relationships and patterns in an organized fashion.

Finally, using the underlying structure and the complex relationships discovered, one can form a “periodic table” of all the 2 X 2 games (131). With this table, one can easily see the games of pure conflict, pure common-interest, and Type games. The periodic table also includes information about the layers and their links, so that one can recognize neighbors of a specific game that may not necessarily be next to it in the table. This table also allows one to see games with dominant strategies and the number of Nash Equilibria that games adjacent to each other might have. Robinson and Goforth compare their periodic table to the scheme presented by Rapoport and Guyer and argue that Rapoport and Guyer’s categories have no particular order to
them or any associations between them. Some games end up in several different “species” (142), and their handling of games is only partial. But Robinson and Goforth’s treatment of the 2 X 2 games “reveals fundamental relationships that will guide research and make the subject more teachable” (143).

The entirety of 2 X 2 games can be represented as one table, separated by layers, slices, tiles, pipes, and hotspots. All of these look at the characteristics of the game behaviorally in terms of how each player will benefit from the game. Most games with similar interest or conflict are often lined up together or clustered somewhere in the table, for example. Not only can one use this table to learn more about conflict and interest, but one can also see possible dominant strategies and the number of Nash Equilibria. They also put a special emphasis on the twelve symmetric games and show that they too are close to each other in the table, which is what I will focus on in the upcoming chapter. Robinson and Goforth’s periodic table of the elements of the 2 X 2 games is a more thoroughly detailed and explained mapping of the 2 X 2 game theory that was started by Rapoport and Guyer. While the latter two only worked with 78 total games, Robinson and Goforth work with 144, so their table is more complex, but makes analysis of the 2 X 2 games much simpler and deeper. With their new, more intricate table, I will be able to see more clearly how the games are connected.
III: Symmetric Games

In the family of games, given by the chart below, first presented in the introduction, one starts with $2 + b, 2 + b; a, 3 - b; 3 - b, a; 1 - a, 1 - a$:

<table>
<thead>
<tr>
<th></th>
<th>X</th>
<th>Y</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>$2 + b, 2 + b$</td>
<td>$a, 3 - b$</td>
</tr>
<tr>
<td>Y</td>
<td>$3 - b, a$</td>
<td>$1 - a, 1 - a$</td>
</tr>
</tbody>
</table>

From here, one can get a two-dimensional graph of the twelve symmetric games by plugging in different values for $a$ and $b$. The graph can be split up into sixteen different sections, each covered by a different symmetric game. The sections are separated by lines with equations given when different payoffs are set equal to each other. So,

\[
\begin{align*}
2 + b &= a & \Rightarrow & & b &= a - 2 \\
2 + b &= 1 - a & \Rightarrow & & b &= -a - 1 \\
a &= 1 - a & \Rightarrow & & a &= -
\end{align*}
\]

\[
\begin{align*}
2 + b &= a - 2 \\
2 + b &= 3 - b & \Rightarrow & & b &= - \\
a &= 3 - b & \Rightarrow & & b &= -a + 3 \\
3 - b &= 1 - a & \Rightarrow & & b &= a + 2
\end{align*}
\]

Graphing these will create the sixteen sections on the a-b graph:

Figure 3- Graph of Sixteen Sections
As stated before, each section of the graph contains a different symmetric game (Appendix 1). The lines that separate each section also contain a different game (Appendix 2); these games occur when switching from one symmetric game with four possible payoffs to another, as described in previous chapters. While the symmetric games all have four preferences per player, the transitional games on the lines, also symmetric, only have three preferences. This transition takes place by switching two consecutive preferences; one can perform $X_{01}$ (meaning one can perform $C_{01}$ and $R_{01}$ at the same time), $X_{12}$, or $X_{23}$ function on any game to get to another neighbor and look at the game that is half way in between the full switch. For example, when performing $X_{01}$, in order to see the transition game, one must look at the game where the 0 and 1 payoffs are both $\neg$. I introduced the path from Prisoner’s Dilemma to Chicken in the first chapter. So, if looking at the change between Chicken and its neighbor, Battle of the Sexes\(^1\), when performing $X_{12}$, one could see that the game, $2$, $2$; $1$, $3$/$3$, $1$; $0$, $0$ goes to $\neg$, $\neg$/$\neg$, $3$/$3$, $\neg$/$\neg$, $0$, $0$. This game has only three preference payoffs. (In my graph, $< - < 3$, which is equivalent to $0 < 1 < 2$. So this game is equivalent to $1$, $1$; $0$, $2$/ $2$, $0$; $0$, $0$.)

From this game to Battle of the Sexes, one sees the final full switch: $1$, $1$; $2$, $3$/$3$, $2$; $0$, $0$.

In places on the large graph where three lines intersect with each other, there are games with only two preference orders. This occurs four times when six different sections of the graph meet. At the points where $a = -$ and $b = -$, $a = -$ and $b = -$; $a = -$ and $b = -$; $a = -$ and $b = -$; and $a = -$ and
b = —, six different sections of the graph meet, and, therefore, at these points, the games that occur only have two preference orders. While the games do not always end up with these values, in my graph, 0 and 1 represent those preference payoffs.

At the point — — , plugging in those numbers for a and b respectively into the equation for the family of games presented, the game outcome is —, —; —, —/ —, —; —, —. This can be seen as — < —, so that is equivalent to 0 < 1. Therefore, the game outcome is 1, 1; 0, 0/ 0, 0; 0, 0. At the point — — , the game outcome is —, —; —, —/ —, —, which is equivalent to 1, 1; 1, 1/ 1, 1; 0, 0. At the point — — , the game outcome is —, —; —, —/ —, —, which is equivalent to 0, 0; 0, 1/ 1, 0, 0. Finally, at the point — — the game outcome is —, —; —, —/ —, —; —, —, which is equivalent to 1, 1; 0, 1/ 0, 1; 1, 1 (Appendix 3). At the point where a = — and b = — on the graph, two of the boundaries meet. At this point, there is an intersection of two lines that occurs where four four-preference games come together. — — also has just two payoff levels.

This total graph, made up of sixteen sections based on the asymptotes created by the lines, does include all twelve four-preference symmetric games. By looking at the graph (Appendix 4), one can see that there are twenty different lines with three possible payoffs; however, two games are repeated twice (when the two same symmetric games meet at 2 different places: Anti-Coordination\textsuperscript{ii} meets with Anti-Stag and Hare\textsuperscript{iii}, and Anti-Chicken\textsuperscript{iv} meets with Anti-Battle of the Sexes\textsuperscript{v}), so, in total, 18 games with three payoffs appear in the graph; this covers all the possible symmetric games with three payoffs.

As stated before, when performing the operations to get from one game to another, I looked at the games halfway in between the full switch of preferences. This helped me to see a
link between multiple symmetric games. Looking at the four games in the middle division of the graph, which includes Prisoner’s Dilemma, Chicken, No Conflict, and Stag and Hare, one can observe how one game turns into another. To get from Prisoner’s Dilemma to Chicken, we have already seen that one can perform an $X_{01}$. To get from Chicken to No Conflict, one must perform an $X_{23}$. The middle transitional game between Chicken and No Conflict gives $-, -, 1, - / -, 1; 0, 0$ (or $2, 2; 1, 2/2, 1; 0, 0$). From No Conflict, one can perform an $X_{01}$ to get to Stag and Hare. This middle game between No Conflict and Stag and Hare is $3, 3; -, 2 / 2; -, -, - (or 2, 2; 0, 1/1, 0; 0, 0)$. And to get back to Prisoner’s Dilemma from Stag and Hare, one must just do an $X_{23}$. The transitional game for this switch is $-, -, 0, - / -, 0, 1; 1, 1$ (or $2, 2; 0, 2/2, 0; 1, 1$).
Going in a circle between these four games starting with Prisoner’s Dilemma going counter clockwise to Chicken first, one can notice that the switches, in order, are $X_{01}$, $X_{23}$, $X_{01}$, and $X_{23}$. So I found that by performing the opposite operation, either $X_{01}$ and $X_{23}$, to each of the middle transition games, I came up with the same game; starting with any of the four transitional games in this part, one can come up with the same second transitional game as any of the others by simply performing $X_{01}$ on a game that originates from $X_{23}$ or $X_{23}$ on a game that originates from $X_{01}$. While the values are not exactly equal, the order graphs are equivalent just by turning fractions into whole numbers using equivalent inequalities, as shown previously.

Figure 6- Transition Games
Changed to Common Game
This is useful because this shows that these four games have a common transitional game, meaning a connection between these four games is implied. One can see that one can get from any four-preference game in this chunk to another one by going through only one other transitional game instead of possibly having to go through two. Formerly, to get from Prisoner’s Dilemma to No Conflict, for example, one could either go from Prisoner’s Dilemma to Stag and Hare to No Conflict or from Prisoner’s Dilemma to Chicken to No Conflict. Using either of these ways would take more work. Now, knowing what the common transitional game is, one can go directly from Prisoner’s Dilemma to No Conflict by looking at only one extra graph. The same can be said about the path to get from Chicken to Stag and Hare or vice versa.

Unlike the middle diamond that appears from the intersection of six different lines, which only has four symmetric games, the other four points, where there are three lines intersecting, have six different games surrounding them, as shown in Appendix 4. So these points involve six symmetric games instead of just four. I had to look at these intersections differently because of the added complexity of two extra games. First, at the point — —, the games that are involved are Coordination\textsuperscript{viii}, Anti-Coordination, Stag and Hare, Prisoner’s Dilemma, Anti-Prisoner’s Dilemma\textsuperscript{ix}, and Anti-Stag and Hare. To go from Anti-Coordination to Coordination, one can perform an $X_{23}$ swap (although, one could also use an $X_{01}$; for this division of the graph,
however, $X_{23}$ is more useful). There is a transition game with the outcomes of $- - 0, 1/ 1, 0; - - (or 2, 2; 0, 1/ 1, 0; 2, 2)$. Moreover, to go from Coordination to Stag and Hare, one must complete an $X_{12}$ swap. The transition game is $3, 3; 0, -/-, 0; -$, $(or 2, 2; 0, 1/ 1, 0; 1, 1)$. Going around the entire piece, there is a pattern of $X_{23}$, then $X_{12}; X_{23}$ is used three times, and $X_{12}$ is also done three times.

Figure 7- First Intersection with Six Games Involved
Contrasting with the point of where the two operations do not include any overlapping preference payoffs, at this point, both operations involve the payoff of 2. This presented a problem as first; nevertheless, after further examination, one can see that by performing the opposite operation on each transition game will again come up with a similar game to which they can all be related. While the players’ choices may be switched sometimes, the points all remain the same. So, for example, performing the equivalent of the X₁₂ operation, an X₁,₂₅ switch, on the transition game between Coordination and Anti-Coordination comes up with the game 0, 0; 0, -/-; -/-, 0. This game could be correspondent to 1, 1; 0, 1/0; 1, 1.

Similarly, performing an X₂₃ operation on the transition game between Coordination and Stag and Hare, which would be exchanging the - and 3, creates 0, -/-; 0, -/-; -/-, 0. This game also corresponds to the game 1, 1; 0, 1/0; 1, 1.
All the transition games with $X_{12}$ swaps can be changed into the same common game by doing the equivalent of an $X_{23}$ swap. Also, all the transition games with an $X_{23}$ swap can be changed into the same game by doing the equivalent of an $X_{12}$ swap. While these two games are slightly different in point value, as seen in the examples provided above, they both have similar graphs. In order to get this game, one must do the swap that is not involved to get to the initial transition game. As with the four symmetric games involved around $-$ $-$, there is a game that all other games must go through to get to another different game.

At $-$ $-$, the transition games between a game with four payoffs and its own anti-game are three-point graphs and the games between a game and something other than its anti-game are four-point graphs (refer to Figure 7). But they all eventually turn into the same three-point graph. In this intersection, the games between Anti-Prisoner’s Dilemma and Prisoner’s Dilemma and Anti-Coordination and Coordination circle back to themselves, meaning they are unique in this section of games. They do, however, lead to the same game. But the game between Stag and Hare and Prisoner’s Dilemma is the same as the game between Anti-Stag and Hare and Anti-Prisoner’s Dilemma, only the players are reversed (they are anti-games), so they too lead to the same graph. Also, the game between Stag and Hard and Coordination is the anti-game of the game between Anti-Stag and Hare and Anti-Coordination. They also lead to the same graph.

At the point $-$ $-$, the games that are involved are Anti-Coordination, Anti-Stag and Hare, Anti-No Conflict, No Conflict, Stag and Hare, and Coordination:
When moving from one game to the next, one must use $X_{01}$ and $X_{12}$ swaps. So in this part of the graph, the payoff of 1 is included in both swaps, meaning again that one cannot perform the two different swaps at the same time as in the part of the graph with the four symmetric games meeting at $- -$. In this case, one must once more perform the opposite operation on each transition game in order to find a common game. So if, for example, one is moving from No Conflict to Stag and Hare, one should perform the $X_{01}$ function to get the middle game of $3, 3; -, 2/2, -; -, -$(or $2, 2; 0, 1/1; 0, 0$). Since the initial operation is $X_{01}$, in
In order to find the middle game that connects this game to the rest of the games in this piece of the graph, one should complete the $X_{12}$ operation, switching - and 2. This game ends up being 3, 3; -,-/ -,-; -,-, which corresponds to 1, 1; 0, 0/ 0, 0; 0, 0.

This game is the same game that one would get by starting with any of the six games surrounding the point – – . Also, as shown in Figure 10, similar to the point – – , the game between Coordination and Anti-Coordination and the game between No Conflict and Anti-No Conflict are unique games, meaning they do not have anti-games appear in this section of the graph. The game between Stag and Hare and Coordination and the game between Anti-Stag and Hare and Anti-Coordination are anti-games, as well as the game between Stag and Hare and No Conflict and the game between Anti-Stag and Hare and Anti-No Conflict. All games, though, can be transformed into the same common game.

At the point – – , the games Anti-No Conflict, Anti-Chicken, Anti-Battle of the Sexes, Battle of the Sexes, Chicken, and No Conflict need either $X_{23}$ or $X_{12}$ to get from one game to the next (in the case of going from Battle of the Sexes to Anti-Battle of the Sexes, one could again use either $X_{23}$ or $X_{01}$, but in this part of the graph, it is more helpful to use $X_{23}$).
Since this is the case, then in order to find the common game that all of the games must go through, one should use $X_{12}$ on the transition games reached by the operation $X_{23}$ and $X_{23}$ on the transition games reached by using $X_{12}$. By performing these operations, one will get the same game each time. The middle game that all the transition games converge to become is a graph with only two points. In one case (moving between Chicken and No Conflict or between...
Anti-Chicken and Anti-No Conflict), performing $X_{12}$ on the transition game, the game is $0, 0; -, -/-, -; -, -$. In the other case (moving between Chicken and Battle of the Sexes or between Anti-Chicken and Anti-Battle of the Sexes), performing $X_{23}$ on the transition game means the game becomes $0, 0; -, -/-, -; -, -$. In both scenarios, the transition game is equivalent to $0, 0; 1, 1/1, 1; 1, 1$. If wanting to reach the middle game from, say, the transition game between Anti-No Conflict and Anti-Chicken, one will perform the equivalent of $X_{12}$ since one originally performs $X_{23}$.

The game between No Conflict and Anti-No Conflict as well as the game between Battle of the Sexes and Anti-Battle of the Sexes, as pictured in Figure 12, again circle back to each other since those games are between a game and its anti-game. The game between Chicken and No Conflict and the game between Anti-Chicken and Anti-No Conflict are identical except for the players; again, they are anti-games, as well as the game between Chicken and Battle of the Sexes and the game between Anti-Chicken and Anti-Battle of the Sexes.

Lastly, at the point $-$ $-$ $-$, Chicken, Battle of the Sexes, Anti-Battle of the Sexes, Anti-Chicken, Anti-Prisoner’s Dilemma, and Prisoner’s Dilemma apply either $X_{01}$ or $X_{12}$ to go from one game to the next. As can be seen in Figure 14, the game between Prisoner’s Dilemma and Anti-Prisoner’s Dilemma, along with the game between Battle of the Sexes and Anti-Battle of
the Sexes, return to themselves when performing the respective operation. The game between Anti-Prisoner’s Dilemma and Anti-Chicken and the game between Prisoner’s Dilemma and Chicken use $X_{12}$ to come up with the game $-, -, 3 / 3, -, -$. Likewise, the game between Anti-Chicken and Anti-Battle of the Sexes and the game between Chicken and Battle of the Sexes use $X_{01}$ to $-, -, 3/3, -, -$. Both of these games are equivalent to $0, 0; 0, 1/1, 0; 0, 0$. 

Figure 14 - Fourth Intersection with Six Games Involved
In order to find the middle game that all converge to, one must perform the opposite operation on each transition game again. For example, starting with the game between Anti-Chicken and Anti-Battle of the Sexes, one will see:

![Figure 15- Transition Game Between Anti-Battle of the Sexes and Anti-Chicken to Common Game](image)

Based on Appendix 4, we already know that my graph contains all the twelve possible symmetric games with four payoffs and all eighteen possible symmetric games with three payoffs. By looking at all the intersections, one can also see that it comprises some of the symmetric games with only two payoffs. For these games, there are seven in total. Four different graphs appear in the intersections. This appears to cover almost all of those seven games. One of the graphs can also be the symmetric game, 1, 1; 0, 1/ 1, 0; 1, 1, or any equivalent of it.

![Figure 16- A Two Outcome Game](image)
The second game can be the symmetric game, 0, 0; 1, 0/0, 1; 0, 0, or any equivalent of it.

![Figure 17- A Second Two Outcome Game](image)

The third game can be the symmetric game, 0, 0; 1, 1/1, 1; 0, 0.

![Figure 18- A Third Two Outcome Game](image)

There are two other possible games with the same points as this graph, but the lines connecting the players’ moves are different. Taking into consideration the lines connecting the players’ move, therefore, the games 0, 0; 1, 1/1, 1; 1, 1 and 1, 1; 0, 0/0, 0; 0, 0 do not show up.

The fourth game can be the symmetric game 0, 0; 1, 0/0, 1; 1, 1.

![Figure 19- A Fourth Two Outcome Game](image)

The game 1, 1; 1, 0/0, 1; 0, 0 has the same points on the graph, but the lines are switched, so the game 1, 1; 1, 0/0, 1; 0, 0 does not appear. So, I have shown that the only symmetric
games that do not appear on the combined graph I have drawn up in Appendix 4 are \(0, 0; 1, 1/1, 1; 1, 1\) and \(1, 1; 0, 0/0, 0; 0, 0\) and \(1, 1; 1, 0/0, 1; 0, 0\).
Battle of the Sexes game is given by 1, 1; 2, 3/ 3, 2; 0, 0.

Anti-Coordination game is given by 2, 2; 0, 1/ 1, 0; 3, 3.

Anti-Stag and Hare game is given by 1, 1; 0, 2/ 2, 0; 3, 3.

Anti-Chicken game is given by 0, 0; 1, 3/ 3, 1; 2, 2.

Anti-Battle of the Sexes game is given by 0, 0; 2, 3/ 3, 2; 0, 0.

No Conflict game is given by 3, 3; 1, 2/ 2, 1; 0, 0.

Stag and Hare game is given by 3, 3; 0, 2/ 2, 0; 1, 1.

Coordination game is given by 3, 3; 0, 1/ 1, 0; 2, 2.

Anti-Prisoner’s Dilemma game is given by 1, 1; 0, 3/ 3, 0; 2, 2.

Anti-No Conflict game is given by 3, 3; 2, 1/ 1, 2; 0, 0.
IV: Conclusion

By looking at the 2 X 2 symmetric games, I have been able to show that there are connections between many of the games that people may not have known about before. I started by counting the number of total 2 X 2 games that exist and explained how one can see the relationships among those. Looking at pipes, hotspots, tiles, layers, and stacks showed that all the games can come together in one common representation. Instead of looking at each individual game along with their neighbors, one can see where each game is in relation to every other possible 2 X 2 game in terms of conflict, specifically. I then turned my focus on the symmetric games to find patterns and associations between them. I proved that, through observing the order graphs of each of the symmetric games, one can come up with a particular graph which shows all of them together. With this graph, separated into sixteen sections, one can find that there are five games that all the games in the sections meet to become. Comparing my results with the total number of 2 X 2 symmetric games with 4 preferences, as well as 3, 2, and 1 preferences, I showed that my results do encompass all of the possible four preference and three preference 2 X 2 symmetric games, as well as most of the two preference games.

2 X 2 games that are discussed regularly in game theory, while only seeming to present a limited amount of information, actually can be explored and dissected to find underlying data that may become crucial to the study of game theory. Looking at the relations between all the 2 X 2 games will add essential understanding to this interesting, growing field.
Appendices
Appendix 1: a-b plane with 4-Preference Symmetric Game Order Graphs
Appendix 2: a-b Plane with 3-Preference Games
(Transition Games between 4-Preference Games)
Order Graphs
Appendix 3: a-b Plane with Graphs at Intersections of Three Lines (Ordinals)
Appendix 4: Graph with All Games
Bibliography

This source confers what makes up a game in game theory. Rapoport and Guyer also define dominance and equilibrium, which are important tools to measure what kind of game a game turns out to be. They classify games based on the state of the equilibrium present in the games. This source was used to learn basic terminology of game theory and to understand what outcomes could occur.

Robinson, David, and David Goforth. *The Topology of the 2x2 Games: a New Periodic Table.*
This source, on which I based most of my research, introduced a new way to count the number of 2 X 2 games that are studied in game theory and organized these games in a way that allows people to understand the games in a new way, based on conflict. Robinson and Goforth also discuss layers, hot spots, slices, and other categories that are formed naturally in their representation of the 2 X 2 games. This source was most helpful during my research process in gathering background information and to describe how game theory works.
Other Sources


This article expands on the information presented in David Robinson and David Goforth’s book, *The Topology of the 2 X 2 Games*. Bryan Bruns provided more specific groupings of games provided by Robinson and Goforth based on payoffs for the players involved in a game. Bruns also showed a revised version of a “periodic table” of the 2 X 2 games that were the focus of my research. This source was most helpful in gaining background information that I could apply to my research and in seeing differing visualizations of the same games that I studied.


This source provided additional information on the original article from Anatol Rapoport and Melvin Guyer. The authors discuss the payoff array that can be presented for each game and how many of the 2 X 2 games are symmetric or not. They count this by considering what payoff preferences each player can possibly have in each game. For example, a player can either have payoffs of 0, 1, 2, or 3, or they can have payoffs of just 0, 1, or 2. Having four different preferences or just three impacts the number of symmetric games that can be made up. This source was beneficial because it provided good background information and allowed me to see how the games could be counted in a new way.


This review shared information about Robinson and Goforth’s book that I used for most of my research. It also provided some points of interest that maybe could have been discussed in further detail. This was helpful because it allowed me to see holes in Robinson and Goforth’s studies, and it allowed me to see where further research could be done.


This source focused on one subset, the Co-operate-Defect games, of 2 X 2 games. Sam Perlo-Freeman provides an analysis of games in which players depend on other players choosing their own dominant strategies. This article was interesting because it allowed me to see how people can make better decisions based on what someone else might do. This article was also helpful because I could examine a subgroup in the Robinson and Goforth model of 2 X 2 games more closely. This provided me with more background information that was helpful while I performed my own original research on the 2 X 2 games.