ON THE DISJUNCTIVE RADO NUMBER
OF THE EQUATIONS

\[ ax_1 = x_2 \ \text{AND} \ bx_1 + x_2 = x_3 \]

by

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Abstract

For a given system of linear equations $L$, the Rado number of the system is the least integer $n$ for which every $t$-coloring of $\{1, \ldots, n\}$ contains a monochromatic solution of one of the equations in $L$, if such an integer exists. In this thesis, the 2-color disjunctive Rado numbers for the equations $ax_1 = x_2$ and $bx_1 + x_2 = x_3$ are determined for more than half of all values of $a$ and $b$. 
1 Introduction

1.1 Ramsey Theory

Ramsey theory is a field of mathematics that first emerged with the work of Frank K. Ramsey, a British mathematician (1903-1930). In general, Ramsey theory studies order that must occur within large random structures. These results come from various disciplines, including number theory, set theory, geometry, theoretical computer science, and ergodic theory. While earlier work, such as that of Schur which will be seen later in this thesis, was later classified within the field of Ramsey theory, Ramsey initiated the study with the statement of his fundamental theorem. Essentially, what Ramsey theory does is show that if a sufficiently large system is partitioned into a finite number of subsystems, there exists at least one subsystem has a particular property [12]. As Walter Deuber summarized the field [1], “Complete disorder is impossible.”

The most common example of Ramsey theory can be situated in terms of people at a party. Given six people, there must be either at least three people who all know each other or at least three people who are (pairwise) strangers. It is known that this conclusion is false for five people. This can be shown by a graph with five vertices. Each vertex represents a person while each edge connecting two vertices represents the relationship between the two individuals. A solid line will represent that the two people know each other, while a dotted line will represent that the two people are strangers.

In order to satisfy the condition, there must be a copy of $K_3$, the complete graph with three vertices, i.e., a triangle, that is completely solid or
completely dotted. As can be seen from the graph, there is no such copy of $K_3$. Therefore, the condition does not hold for five people.

To show that the condition does hold for six people, look at any one vertex (colored grey in this graphic). By the pigeon hole principle, since there are five edges stemming from this vertex, there must be either three or more dotted edges or three or more solid edges. Without loss of generality, let these edges be solid.

The three other vertices are all connected by edges. If they are all dotted (as in the graphic), then this shows that there are three people that do not know each other. If they are not all dotted, then at least one of them must be solid. This solid line, with two of the solid edges connected to $v$, will form a solid copy of $K_3$. Therefore, there will always be a copy of $K_3$ whose edges are completely dotted or completely solid.

As stated earlier, there are many different problems that fall under the umbrella of Ramsey theory. A result in number theory discovered by Bartel Leendert van der Waerden, although completed before Ramsey’s work, is considered Ramsey theory. Later, Paul Erdős and George Szekeres applied Ramsey’s theorem to geometry, completing the first work in Euclidean Ramsey theory. Other applications of Ramsey theory include results in algebra, topology, set theory, logic, ergodic theory, and computer science, among other fields. In this thesis, an application of Ramsey theory within the field of number theory will be examined.
1.2 Schur Numbers

A function $\Delta : \{1, \ldots, n\} \to \{0, \ldots, t - 1\}$ can be considered a $t$-coloring of the set $\{1, \ldots, n\}$. Given a $t$-coloring $\Delta$ and a linear equation in $m$ variables, a solution $(x_1, \ldots, x_m)$ is monochromatic if

$$\Delta(x_1) = \Delta(x_2) = \cdots = \Delta(x_m).$$

In 1916, Issai Schur was able to prove that for all $t \geq 2$, there exists a least integer $n$ for which there is a monochromatic solution to the equation $x_1 + x_2 = x_3$ for every $t$-coloring of $\{1, \ldots, n\}$, although he did not actually specify the value of $n$, instead writing only an existence proof. These integers for each respective $t$-coloring are referred to as the Schur numbers [15]. It is known that the Schur number for a 2-coloring is 5, shown as follows.

- Without loss of generality, let $\Delta(1) = 0$.
- Assume $\Delta(2) = 0$. Then the solution $(x_1, x_2, x_3) = (1, 1, 2)$ would be monochromatic. Therefore, $\Delta(2) = 1$.
- Next, assume $\Delta(4) = 1$. Then the solution $(2, 2, 4)$ would be monochromatic. Therefore, $\Delta(4) = 0$.
- Now assume $\Delta(3) = 0$. Then the solution $(1, 3, 4)$ would be monochromatic. Therefore, $\Delta(3) = 1$.
- Finally, assume $\Delta(5) = 0$. Then the solution $(1, 4, 5)$ would be monochromatic, all “color” 0. Therefore, $\Delta(5) = 1$. But then the solution $(2, 3, 5)$ would be monochromatic, all color 1.

Since either coloring of 5 leads to a monochromatic solution, the Schur number for a 2-coloring is at most 5. It must then be shown that there exists a coloring of $\{1, \ldots, 4\}$ such that there is no monochromatic solution to the equation $x_1 + x_2 = x_3$. This coloring will be that which was found above; that is, $\Delta(1) = \Delta(4) = 0$ and $\Delta(2) = \Delta(3) = 1$. On this interval, there are only four solutions to this equation. These are $(x_1, x_2, x_3) = (1, 1, 2), (1, 2, 3), (1, 3, 4), \text{ and } (2, 2, 4)$.
• The solutions $(1, 1, 2)$ and $(1, 2, 3)$ are not monochromatic since $\Delta(1) = 0$ and $\Delta(2) = 1$.

• The solution $(1, 3, 4)$ is not monochromatic since $\Delta(1) = 0$ and $\Delta(3) = 1$.

• The solution $(2, 2, 4)$ is not monochromatic since $\Delta(2) = 1$ and $\Delta(4) = 1$.

Therefore, since there is a coloring of $\{1, \ldots, 4\}$ that does not contain a monochromatic solution to $x_1 + x_2 = x_3$, the Schur number for a 2-coloring must be greater than 4. Hence, the Schur number for a 2-coloring is 5. Currently, only two other Schur numbers are known; 14 for a 3-coloring and 45 for a 4-coloring [16].

Henceforth, instead of using this drawn out method of showing that a particular number is forced to be a color, a shorter method will be used. To illustrate in the previous example, to show that $\Delta(2) = 1$ is forced, it will be stated that $1 + 1 = 2$. Since it is already known that $\Delta(1) = 0$, it is known that the value $\Delta(2) = 0$ would cause a monochromatic solution, $\Delta(2) = 1$ is necessary.

1.3 Rado Numbers

In 1933, Richard Rado expanded the concept developed by Schur to systems of linear equations. In doing so, Rado was able to develop necessary and sufficient conditions to determine if an arbitrary system of linear equations has a monochromatic solution for a $t$-coloring of $\{1, \ldots, n\}$. For a given system of linear equations $L$, the least integer $n$ for which every $t$-coloring of $\{1, \ldots, n\}$ contains a monochromatic solution of one of the equations in $L$, if such an integer exists, is called the Rado number of the system. If such an integer does not exist, then the $t$-color Rado number for that system is said to be infinite [1, 9, 10, 11].

There has been much work done on finding families of solutions for this problem, primarily with 2-color Rado numbers of systems containing a single linear equation. Because most of the work that has been done in this area has been done in regard to 2-color Rado numbers and the topic of this thesis will also be in two colors, it will henceforth be assumed that the term Rado number means 2-color Rado number. A major goal of research into
Rado numbers is to find the form of a Rado number for a completely general linear equation. A significant step towards that goal was analyzing an equation of the form $a_1x_1 + \cdots + a_mx_m = x_0$, where $a_1, \ldots, a_m$ are positive integer constants. The Rado number of this equation was conjectured to be $a(a + b)^2 + b$, where $a = \min \{a_1, \ldots, a_m\}$ and $b = \sum_{i=1}^{m} a_i - a$ by Brian Hopkins and Daniel Schaal, who proved the $a = 2$ case in 2005 [3]. The full conjecture was later proven by Song Guo and Zhi-Wei Sun in 2008 [2].

1.4 Disjunctive Rado Numbers

Out of the Rado number problem arose a bevy of variations, one of which is the subject of this thesis. For two equations $L_1$ and $L_2$, the disjunctive Rado number is the least integer $n$ for which every 2-coloring of $\{1, \ldots, n\}$ contains a monochromatic solution of $L_1$ or a monochromatic solution of $L_2$, if such an integer exists. If such an integer does not exist, then the disjunctive Rado number is said to be infinite for that set of equations. Although the quantity of work on disjunctive Rado numbers is not nearly as large as that of standard Rado numbers, there still has been significant progress made.

The concept of disjunctive Rado numbers was first introduced by Brenda Johnson and Schaal [4] in work that was first published in 2005. This thesis included results of two different pairs of equations. First, Johnson and Schaal found the form of a disjunctive Rado number of the equations $x_1 + a = x_2$ and $x_1 + b = x_2$ to be

$$\begin{cases} a + b + 1 - \gcd(a, b) & \text{if } \frac{a}{\gcd(a,b)} + \frac{b}{\gcd(a,b)} \text{ is odd,} \\ \infty & \text{otherwise.} \end{cases}$$

The disjunctive Rado number for the second set of equations, those of the form $ax_1 = x_2$ and $bx_1 = x_2$, is slightly more involved. Johnson and Schaal determined that for all integers $a > 1$ and $b > 1$, if there exist natural numbers $c, s, \text{ and } t$ such that $a = c^s$ and $b = c^t$, where $c$ is the largest such integer, and $s + t$ is an odd integer, then the disjunctive Rado number is $c^{s+t-1}$. For all other integers, $a > 1$ and $b > 1$, the Rado number is infinite.

In 2003, Wojciech Kosek and Schaal laid claim to the first paper published in the study of disjunctive Rado numbers [6]. They cite a pre-print version of Johnson and Schaal as background, even though that paper was not published until two years later. Kosek and Schaal found the disjunctive Rado number for linear equations of the form $\sum_{i=1}^{m-1} x_i = x_m$ and $\sum_{i=1}^{n-1} x_i = x_n$,
where $3 \leq m \leq n$, to be

$$\begin{cases} m^2 - m - 1 & \text{for } m \leq n \leq m + 1, \\ m^2 - 2m - 1 & \text{for } m + 2 \leq n \leq m^2 - 2m + 2, \\ n - 1 & \text{for } m^2 - 2m + 3 \leq n \leq m^2 - m - 1, \\ m^2 - m - 1 & \text{for } n \geq m^2 - m. \\ \end{cases}$$

In 2007, Dusty Sabo, Schaal, and Jacent Tokaz, released a paper that expanded the first result from Johnson and Schaal and adding another variable, considering the equations $x_1 + x_2 + a = x_3$ and $x_1 + x_2 + b = x_3$ [14]. They found that the disjunctive Rado number is

$$\begin{cases} 4a + 5 & \text{if } a \leq b \leq a + 1, \\ 3a + 4 & \text{if } a + 2 \leq b \leq 3a + 2, \\ b + 2 & \text{if } 3a + 3 \leq b \leq 4a + 2, \\ 4a + 5 & \text{if } 4a + 3 \leq b. \\ \end{cases}$$

In 2008, Darren Row looked at equations similar to those examined by Kosek and Schaal five years prior, but added a constant into each equation, resulting in the equations $x_1 + \cdots + x_{m-1} + a = x_m$ and $x_1 + \cdots + x_{n-1} + b = x_n$, for all integers $a, b, m, n$ that satisfy $3 \leq m < n, a \geq 0, b \geq 0$ [13]. The result of his work was that here the disjunctive Rado number is

$$\begin{cases} m^2 + m (a - 2) + 1 & \text{if } m + a + 1 \leq n + b \leq m^2 + m (a - 2) + 2, \\ n + b + 1 & \text{if } m^2 + m (a - 2) + 3 \leq n + b \leq m^2 + m (a - 1) + a - 1, \\ m^2 + m (a - 1) + a - 1 & \text{if } m^2 + m (a - 1) + a - 1 \leq n + b. \\ \end{cases}$$

The most recent work done in the field of disjunctive Rado numbers was a thesis by Liz Lane-Harvard [7], who looked at equations in the form $ax_1 + x_2 = x_3$ and $bx_1 + x_2 = x_3$, for all integers $a$ and $b$ that satisfy $2 \leq a \leq b$. This thesis was recently reworked into a paper and submitted for publication by Lane-Harvard and Schaal [8]. Due to the complexity of the result (13 cases), it is not given here. Many of her methods, though, will be adapted for the purposes of this thesis.
1.5 Overview of New Results

The goal of this thesis is to identify patterns of the disjunctive Rado numbers for the equations \(ax_1 = x_2\) and \(bx_1 + x_2 = x_3\). This problem presents a particular challenge because, unlike previously proven results, the two equations are not symmetric; that is, the two equations differ by more than merely a coefficient or constant. At this point, the disjunctive Rado number has been found for only certain \(a\) and \(b\), specified below, more than half of all cases. Table 1 gives the disjunctive Rado numbers for \(a \leq 15\) and \(b \leq 15\). The italicized values are those not covered by the results of this thesis. These were found using Mathematica, details are given in the Appendix.

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In this thesis, the disjunctive Rado number for various cases of \(a\) and \(b\) will be determined. In four theorems, the results for all cases where \(a > b - 1\) are proven. These results include the lower left half of Table 1 through the first superdiagonal. Two additional cases are proven: \(b = a^2 - 1\) and \(b = 2a\) for \(a\) even. Examples of these infinite families occur a few times in the upper right section of Table 1.

The first case occurs when \(a = b + 1\) or \(a = b + 2\), which are represented by the two immediate subdiagonals in the table. Next, the case \(a > b + 2\) will be proven, which fills out the lower left triangle of the table. Third, the disjunctive Rado number for these two equations when \(a = b\) will be shown.
This corresponds to the table’s diagonal. Then, the case $a = b - 1$ when $b \geq 4$ will be shown, which can be found in the first superdiagonal in the table. Next, the pattern when $b = a^2 - 1$ will be proven to have a consistent disjunctive Rado number. This occurs only three times in this chart, at $b = 3$, $b = 8$, and $b = 15$, and occurs more sparingly as $a$ and $b$ increase. Finally, the case where $b = 2a$ when $a$ is even and $a \geq 4$ will be shown. This case will also occur only sparingly, but will happen at every other instance of $a$ in a somewhat flat diagonal.

2 Background Results

Before presenting the new contributions made in this thesis, there is an earlier result that we will need. This is not a result of a disjunctive Rado number; rather, it is a proof of a forced coloring of the equation $bx_1 + x_2 = x_3$. Scott Jones and Schaal proved that the Rado number for $bx_1 + x_2 = x_3$ is $b^2 + 3b + 1$ [5]. In [7], Lane-Harvard shows that not only is this the Rado number for this equation, but also that

$$\Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq b, \\
1 & \text{if } b + 1 \leq x \leq b^2 + 2b, \\
0 & \text{if } b^2 + 2b + 1 \leq x \leq b^2 + 3b,
\end{cases}$$

is forced. This coloring will henceforth be referred to as the stubborn coloring. Lane-Harvard begins with a lemma that accounts for part of the coloring and then finishes the result with a later proposition [7]. We recount her work here because this result was put forth in her thesis, which is not widely available (at least until the publication of [8]), and we are using a modified version of her result.

**Lemma 1.** For every integer $b \geq 2$, the coloring $\Delta$ on $\{1, \ldots, b^2 + 2b + 1\}$ given by

$$\Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq b \text{ and } x = b^2 + 2b + 1, \\
1 & \text{if } b + 1 \leq x \leq b^2 + b + 1,
\end{cases}$$

has no monochromatic solutions to the equation $bx_1 + x_2 = x_3$.

**Proof.** Let $b$ be an integer such that $b \geq 2$. Suppose $\Delta$ does not have a monochromatic solution to the equation $bx_1 + x_2 = x_3$ and, without loss of
generality, assume $\Delta (1) = 0$. Then,

$$b \cdot 1 + 1 = b + 1,$$

meaning $\Delta (b + 1) = 1$. From this follows

$$b \cdot (b + 1) + b + 1 = b^2 + 2b + 1,$$

meaning $\Delta (b^2 + 2b + 1) = 0$. Suppose that $\Delta (b) = 1$. Then, since

$$b \cdot b + 1 = b^2 + b + 1,$$

$\Delta (b^2 + b + 1) = 0$. But then,

$$b \cdot 1 + b^2 + b + 1 = b^2 + 2b + 1,$$

and $\Delta (b^2 + 2b + 1) = 1$, which contradicts the earlier assertion that $\Delta (b^2 + 2b + 1) = 0$. Therefore, $\Delta (b) = 0$.

By induction, it will be shown that $\Delta (b^2 + (1 - k) b + 1) = 1$ and $\Delta (b - k) = 0$, for $k \in \{1, \ldots, b - 1\}$.

For $k = 1$,

$$b \cdot b + 1 = b^2 + 1,$$

meaning that $\Delta (b^2 + 1) = 1$. Then,

$$b \cdot (b - 1) + b + 1 = b^2 + 1,$$

meaning $\Delta (b - 1) = 0$.

For the induction step, let $k_0 \in \{1, \ldots, b - 1\}$ and assume $\Delta (b^2 + (1 - k_0) b + 1) = 1$ and $\Delta (b - k_0) = 0$. It must be shown that

$$\Delta (b^2 + (1 - (k_0 + 1)) b + 1) = 1$$

and

$$\Delta (b - (k_0 + 1)) = 0.$$

Since

$$b^2 + (1 - (k_0 + 1)) b + 1 = b^2 - k_0 b + 1$$

and

$$b (b - k_0) + 1 = b^2 + k_0 b + 1,$$
then $\Delta (b^2 - k_0 b + 1) = 1$. Finally,

$$b \cdot (b - (k_0 + 1)) + b + 1 = b^2 + k_0 b + 1,$$

meaning that $\Delta (b - (k_0 + 1)) = 0$. Therefore, $\Delta (b^2 + (1 - k_0) b + 1) = 1$ and $\Delta (b - k_0)$, for $k_0 = 1, \ldots, b - 2$.

It has now been shown that $\Delta (x) = 0$ for $1 \leq x \leq b$ and $x = b^2 + 2b + 1$. Since this has been shown, it follows that $\Delta (x) = 1$ for $b + 1 \leq x \leq b^2 + b + 1$. From the induction step, $\Delta (x) = 1$ for both $b + 1 \leq x \leq b^2 - b + 1$ and $b^2 - b + 2 \leq x \leq b^2 + b$. Since $\Delta (x) = 0$, the second part of the statement can be arrived at using the same process that was previously used to show that $\Delta (b + 1) = 1$. Finally, since $b \cdot 1 + b^2 + b + 1 = b^2 + 2b + 1$, we have $\Delta (b^2 + b + 1) = 1$. \hfill \Box

Lane-Harvard continues with a proposition that completes the aforementioned stubborn coloring.

**Proposition 1.** For every $p \in \{0, \ldots, b - 1\}$ for $b \geq 2$, every coloring $\Delta$ on $\{1, \ldots, b^2 + 2b + 1 + p\}$ that does not contain a monochromatic solution to the equation $bx_1 + x_2 = x_3$ satisfies

$$\Delta (x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq b, \\
1 & \text{if } b + 1 \leq x \leq b^2 + b + 1 + p, \\
0 & \text{if } b^2 + 2b + 1 \leq x \leq b^2 + 2b + 1 + p. 
\end{cases}$$

**Proof.** Let $b \geq 2$ be given, let $p \in \{0, \ldots, b - 1\}$, and assume the coloring $\Delta$ on $\{1, \ldots, b^2 + 2b + 1 + p\}$ does not contain a monochromatic solution to the equation $bx_1 + x_2 = x_3$. In particular, this coloring $\Delta$ does not have a monochromatic solution to the equation $bx_1 + x_2 = x_3$ in the domain $\{1, \ldots, b^2 + 2b + 1\}$, so Lemma 1 applies. Therefore, we only need to check that $\Delta (x) = 1$ for $b^2 + b + 1 < x \leq b^2 + b + 1 + p$ and $\Delta (x) = 0$ for $b^2 + 2b + 1 \leq x \leq b^2 + 2b + 1 + p$. For every $j \in \{0, \ldots, p\}$, it is known that

$$\Delta (b + 1) = \Delta (b + 1 + j) = 1,$$

meaning that

$$b \cdot (b + 1) + b + 1 + j = b^2 + 2b + 1 + j$$

forces $\Delta (b^2 + 2b + 1 + j) = 0$. 

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Now, for all \( j \in \{0, \ldots, p\} \),
\[
b \cdot 1 + b^2 + b + 1 + j = b^2 + 2b + 1 + j
\]
forces \( \Delta(b^2 + b + 1 + j) = 0 \).

Therefore, every coloring \( \Delta \) on \( \{1, \ldots, b^2 + 2b + 1 + p\} \) that does not produce a monochromatic solution of \( bx_1 + x_2 = x_3 \) satisfies
\[
\Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq b, \\
1 & \text{if } b + 1 \leq x \leq b^2 + b + 1 + p, \\
0 & \text{if } b^2 + 2b + 1 \leq x \leq b^2 + 2b + 1 + p. 
\end{cases}
\]

From here, in order to get the stubborn coloring given in Equation 5, note that this equation is satisfied for all \( p \in \{0, \ldots, b - 1\} \). Therefore, by letting \( p = b - 1 \), Equation 7 becomes Equation 5. This result of Lane-Harvard’s will be instrumental in proving theorems about the disjunctive Rado number for the equations \( ax_1 = x_2 \) and \( bx_1 + x_2 = x_3 \).

The Rado number for the equation \( bx_1 + x_2 = x_3 \) is defined here, along with the coloring that it forces, because it supplies an upper bound for the disjunctive Rado number being studied and the resulting coloring will play a vital role in searching for the disjunctive Rado number that is the main topic of this thesis. However, the equation \( ax_1 = x_2 \) will obviously also play a vital role, but its Rado number has not yet been defined.

It is relatively simple to show that the Rado number for the equation \( ax_1 = x_2 \) is infinity. First, without a loss of generality, let \( \Delta(1) = 0 \). Then, since \( a \cdot 1 = a \), \( \Delta(a) = 1 \). Then, \( a \cdot a = a^2 \), meaning \( \Delta(a^2) = 0 \). Next, \( a \cdot a^2 = a^3 \), which forces \( \Delta(a^3) = 1 \). This pattern will continue on, creating a forced coloring of
\[
\Delta(a^x) = \begin{cases} 
0 & \text{if } x \text{ is even}, \\
1 & \text{if } x \text{ is odd.}
\end{cases}
\]

Since only powers of \( a \) will be forced, and none of these integers will be forced to be color 0 and 1 (since an integer cannot be even and odd at the same time), a monochromatic solution to \( ax_1 = x_2 \) can always be avoided.
3 New Results

Now that the stubborn coloring has been proven to be a forced coloring for the equation \(bx_1 + x_2 = x_3\), many of the aforementioned results for the disjunctive Rado number of \(ax_1 = x_2\) and \(bx_1 + x_2 = x_3\) can be proven.

Theorem 1 (First and Second Subdiagonals). When \(a = b + 1\) or \(a = b + 2\), the disjunctive Rado number is \(b^2 + 3b + 1\).

Proof. First, it must be shown that there always will be a monochromatic solution to either \(ax_1 = x_2\) or \(bx_1 + x_2 = x_3\). By the result in [2], it is known that the Rado number for \(bx_1 + x_2 = x_3\) is \(b^2 + 3b + 1\), meaning that the disjunctive Rado number must be less than or equal to \(b^2 + 3b + 1\).

Second, it must be shown that there exists a coloring \(\Delta\) on \(\{1, \ldots, b^2 + 3b\}\) for which is not a monochromatic solution of either equation. Let \(\Delta\) be the stubborn coloring. Since the stubborn coloring produces no monochromatic solutions to \(bx_1 + x_2 = x_3\) in \(\{1, \ldots, b^2 + 3b\}\) by Theorem 1, the only equation that must be considered is \(ax_1 = x_2\).

As these cases are being examined, keep in mind that we are looking to show that there will not be a monochromatic solution to \(ax_1 = x_2\). As such, we only have to eliminate cases in which \(\Delta(x_1) = \Delta(x_2)\). Therefore, we are only analyzing cases where both \(x_1\) and \(x_2\) are in \(\{1, \ldots, b\}\) or \(\{b^2 + b + 1, \ldots, b^2 + 3b\}\) or both \(x_1\) and \(x_2\) are in \(\{b + 1, \ldots, b^2 + b\}\).

There are no monochromatic solutions to \(ax_1 = x_2\) where \(x_1, x_2 \in \{1, \ldots, b\}\) because

\[
a \cdot 1 = a > b.
\]

There are no monochromatic solutions to the equation \(ax_1 = x_2\) using \(x_1 \in \{1, \ldots, b\}\) and \(x_2 \in \{b^2 + 2b + 1, \ldots, b^2 + 3b\}\) because

\[
a \cdot b = ab \leq b(b + 2) < b^2 + 2b + 1
\]

by assumption on \(a\). There is not a monochromatic solution to the equation \(ax_1 = x_2\) where \(x_1, x_2 \in \{b + 1, \ldots, b^2 + 2b\}\) because

\[
a \cdot (b + 1) \geq b^2 + 2b + 1.
\]

There is not a monochromatic solution to the equation \(ax_1 = x_2\) where \(x_1, x_2 \in \{b^2 + 2b + 1, \ldots, b^2 + 3b\}\) because

\[
a \cdot (b^2 + 2b + 1) \geq b^3 + 3b^2 + 3b + 1 > b^2 + 3b.
\]
Finally, there are no monochromatic solutions to the equation \( ax_1 = x_2 \) with \( x_1 \in \{ b^2 + 2b + 1, \ldots, b^2 + 3b \} \) and \( x_2 \in \{ 1, \ldots, b \} \) because
\[
a \cdot (b^2 + 2b + 1) > b.
\]
Since these are the only five instances in which a monochromatic solution could arise, and it has been shown that there will never be a monochromatic solution in any of these cases, the stubborn coloring will not have any monochromatic solutions to the equations \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \).

**Theorem 2** (Lower Left Triangle). When \( a > b + 2 \), the disjunctive Rado number is
\[
\begin{cases}
  ka & \text{if } b^2 + 2b + 1 \leq ka \leq b^2 + 3b, \text{ for some integer } k, \\
  b^2 + 3b + 1 & \text{otherwise}.
\end{cases}
\]

**Proof.** Case 1: \( b^2 + 2b < ka < b^2 + 3b + 1 \), for some integer \( k \).

First, it must be shown that there will always be a monochromatic solution of either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \). Due to Proposition 1, it is known that the stubborn coloring is forced under the equation \( bx_1 + x_2 = x_3 \). Therefore, it is known that \( \Delta(1) = \cdots = \Delta(b) = 0 \) and \( \Delta(b^2 + 2b + 1) = \cdots = \Delta(b^2 + 3b) = 0 \). Therefore, \( \Delta(ka) = 0 \). Also, by the condition of the theorem, \( b^2 + 2b + 1 \leq ka \leq b^2 + 3b \), meaning \( ka \leq b(b + 3) \). Since \( a \geq b + 3 \), we have \( k \leq b \). Therefore, \( \Delta(k) = 0 \) and \( \Delta(ka) = 1 \) is forced by the equation \( ax_1 = x_2 \) when \( x_1 = k \) since \( \Delta(1) = \cdots = \Delta(b) = 0 \). Hence, there will always be a monochromatic solution to either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \).

Second, it must be shown that there is a coloring \( \Delta \) on \( \{ 1, \ldots, ka - 1 \} \) for which there exists no monochromatic solution of either equation. Since the stubborn coloring is a good coloring of \( \{ 1, \ldots, b^2 + 3b \} \) for the equation \( bx_1 + x_2 = x_3 \) and \( ka < b^2 + 3b \), the only thing that needs consideration is the effect of \( ax_1 = x_2 \) on this coloring.

Since \( a > b \), the coloring \( \Delta \) on \( \{ 1, \ldots, b \} \) will not be further constrained by the equation \( ax_1 = x_2 \). Also, the smallest integer that could be arrived at by using \( ax_1 = x_2 \) and letting \( x_1 \) be color 1 is
\[
(b + 1)(b + 1) = b^2 + 2b + 1.
\]
Therefore, \( \Delta \) on \( \{ b + 1, \ldots, b^2 + 2b \} \) will not have a monochromatic solution \( ax_1 = x_2 \). Finally, \( \Delta \) on \( \{ b^2 + 2b + 1, \ldots, b^2 + 3b \} \) will not be further constrained by \( ax_1 = x_2 \), but only at \( ka \) since there are only \( b \) numbers in this
set. Since \( a > b + 2 \), only one multiple of \( a \) can exist between \( b^2 + 2b + 1 \) and \( b^2 + 3b \). Therefore, \( ka \) will be the first number at which a monochromatic solution will occur, since the equation \( ax_1 = x_2 \) will force \( \Delta(ka) = 1 \) while the equation \( bx_1 + x_2 = x_3 \) will force \( \Delta(ka) = 0 \).

Case 2: There is no integer \( k \) such that \( b^2 + 2b < ka < b^2 + 3b + 1 \).

Since the disjunctive Rado number in this case is \( b^2 + 3b + 1 \), it is known that the Rado number of \( bx_1 + x_2 = x_3 \) is \( b^2 + 3b + 1 \), meaning that there will always be a monochromatic solution forced at \( b^2 + 3b + 1 \).

Next, it must be shown that there is a coloring \( \Delta \) on \( \{1, \ldots, b^2 + 3b\} \) for which there is no monochromatic solution to either \( ax_1 = x_2 \) and \( bx_1 + x_2 = x_3 \). Assume that this coloring is the stubborn coloring. Due to Proposition 1, it is known that there are no monochromatic solutions to the equation \( bx_1 + x_2 = x_3 \). Therefore, it only needs to be shown that the equation \( ax_1 = x_2 \) does not put further constraints on the stubborn coloring.

By the same reasoning used in Case 1 of this proof, \( \Delta \) on \( \{1, \ldots, b\} \) and \( \{b + 1, \ldots, b^2 + 2b\} \) will not have monochromatic solution to the equation \( ax_1 = x_2 \). Now, since there is no integer \( k \) such that \( b^2 + 2b < ka < b^2 + 3b + 1 \), the equation \( ax_1 = x_2 \) does not put any further constraints on the stubborn coloring. Therefore, the stubborn coloring does not have any monochromatic solutions to the equation \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \). \( \square \)

**Theorem 3** (Diagonal). *When \( a = b \), the disjunctive Rado number is \( b(b+1) \).*

**Proof.** Let \( \Delta(1) = 0 \). Then,

\[
 a \cdot 1 = a = b,
\]

meaning \( \Delta(b) = 1 \). Also,

\[
 b \cdot 1 + 1 = b + 1,
\]

meaning \( \Delta(b + 1) = 1 \). Then,

\[
 a \cdot b = ab = b^2,
\]

meaning \( \Delta(b^2) = 0 \). Finally,

\[
 b \cdot 1 + b^2 = b^2 + b
\]

forces \( \Delta(b^2 + b) = 1 \), but

\[
 a \cdot (b + 1) = ab + a = b^2 + b
\]

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forces \( \Delta(b^2 + b) = 0 \). Therefore, there will always be a monochromatic solution to one of the equations at \( b^2 + b \).

Next, it must be shown that there exists a coloring of \( \{1, \ldots, b^2 + b - 1\} \) such that there is not a monochromatic solution to the equation \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \). Let \( \Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq b - 1, \\
1 & \text{if } b \leq x \leq b^2 - 1, \\
0 & \text{if } b^2 \leq x \leq b^2 + b - 1 
\end{cases} \)

be a coloring of \( \{1, \ldots, b^2 + b - 1\} \). Using the numbers \( \{1, \ldots, b - 1\} \) on the left-hand side of each equation, the smallest number that can result is

\[
a \cdot 1 = a = b
\]

from the equation \( ax_1 = x_2 \) and the largest number that can result is

\[
b \cdot 1 + 1 = b + 1
\]

from the equation \( bx_1 + x_2 = x_3 \). The largest number that can result from the use of these numbers on the left-hand side of each equation is

\[
a \cdot (b - 1) = ab - a = b^2 - b
\]

from the equation \( ax_1 = x_2 \) and the largest number that can result from the equation \( bx_1 + x_2 = x_3 \) is

\[
b \cdot (b - 1) + b - 1 = b^2 - 1.
\]

Therefore, since the numbers \( \{1, \ldots, b - 1\} \) can only produce the numbers \( \{b, \ldots, b^2 - 1\} \), there can be no monochromatic solutions using only these numbers on the left-hand side.

\[\square\]

**Theorem 4** (First Superdiagonal). When \( b = a + 1 \), the disjunctive Rado number is at least \( a^2 \).

*Proof.* It must be shown that there is a coloring of \( \{1, \ldots, a^2 - 1\} \) such that there is no monochromatic solution to either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \). Let

\[
\Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x \leq a - 1, \\
1 & \text{if } x = a, \\
0 & \text{if } x = a + 1, \\
1 & \text{if } a + 2 \leq x \leq 2a, \\
0 & \text{if } x = 2a + 1, \\
1 & \text{if } 2a + 2 \leq x \leq a^2 - 1.
\end{cases}
\]
First, the coloring will be analyzed with regards to the equation $ax_1 = x_2$. It is known that $x_1 < a$ because

$$a \cdot a = a^2,$$

which is above the range of the coloring. Therefore, $x_1$ must lie in \{1, \ldots, a - 1\}, meaning that only the cases where this occurs and $\Delta(x_2) = 0$ must be examined further. Furthermore, it is known that $x_2 \geq a$ because

$$a \cdot 1 = a.$$

Because of this and the earlier constraint placed on $x_1$, the only cases that need to be analyzed further are when $x_2 = a + 1$ or $x_2 = 2a + 1$. First, the case where $x_1 \in \{1, \ldots, a - 1\}$ and $x_2 = a + 1$ cannot occur because

$$a \cdot 1 = a < a + 1$$

and

$$a \cdot 2 = 2a > a + 1.$$ 

Therefore, this case will not occur. Next, the case where $x_1 \in \{1, \ldots, a - 1\}$ and $x_2 = 2a + 1$ cannot occur because

$$a \cdot 2 = 2a < 2a + 1$$

and

$$a \cdot 3 = 3a > 2a + 1.$$ 

Therefore, this case will not occur. Since both cases that were not previously eliminated have been shown to be impossible, this coloring will not produce a monochromatic solution to $ax_1 = x_2$.

Next, the coloring will be analyzed with regards to the equation $bx_1 + x_2 = x_3$. It is known that $x_1 < a - 1$ because

$$b(a - 1) + 1 = (a + 1)(a - 1) + 1 = a^2,$$

which is above the range of the coloring. Therefore, $x_1$ must lie in \{1, \ldots, a - 2\}, meaning that only the cases where this occurs and $\Delta(x_2) = \Delta(x_3) = 0$ must be examined further. Furthermore, it is known that $x_3 \geq a + 2$ because

$$b \cdot 1 + 1 = b + 1 = a + 2.$$ 

Because of this and the earlier constraint placed on $x_1$, the only cases that need to be analyzed further are when $x_3 = 2a + 1$. 

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• First, the case where $x_1 \in \{1, \ldots, a-2\}$, $x_2 \in \{1, \ldots, a-1\}$, and $x_3 = 2a + 1$ will be examined. When $x_1 = 1$,

$$b \cdot 1 + x_2 = (a + 1) \cdot 1 + x_2 = 2a + 1$$

results in $x_2 = a$, which is above the given range of $x_2$. When $x_1 = 2$,

$$b \cdot 2 + x_2 = (a + 1) \cdot 2 + x_2 = 2a + 1$$

results in $x_2 = -1$, which clearly cannot occur. When $x_1 > 2$, $x_2$ continues to decrease, thus remaining negative. Since $x_2$ must either be $a$ or negative, it will never fall in the range $\{1, \ldots, a-1\}$, which contradicts the earlier assumption.

• Next, the case where $x_1 \in \{1, \ldots, a-2\}$, $x_2 = a + 1$, and $x_3 = 2a + 1$ will be examined. Since

$$b \cdot 1 + a + 1 = 2a + 2 > 2a + 1,$$

$x_2$ cannot be $a + 1$, which contradicts the earlier assumption.

• Finally, the case where $x_1 \in \{1, \ldots, a-1\}$, $x_2 = 2a+1$, and $x_3 = 2a+1$ will be examined. Since

$$b \cdot 1 + 2a + 1 = 3a + 2 > 2a + 1,$$

$x_2$ cannot be $2a + 1$, which contradicts the earlier assumption.

Therefore, since the cases that were not already eliminated have been shown to be impossible, this coloring will not result in a monochromatic solution to $bx_1 + x_2 = x_3$.  

While it has been proven that the lower bound of the disjunctive Rado number is $a^2$, we conjecture that this is in fact the disjunctive Rado number. This conjecture is supported by the data in Table 1.

**Theorem 5 (Square Case).** When $b = a^2 - 1$, the disjunctive Rado number is $a^2$.

**Proof.** First, it must be shown that there will always be a monochromatic solution to either $ax_1 = x_2$ or $bx_1 + x_2 = x_3$ at $a^2$. Without a loss of generality, let $\Delta(1) = 0$. Then,

$$a \cdot 1 = a,$$

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which means that $\Delta(a) = 1$. Next,

$$a \cdot a = a^2,$$

which forces $\Delta(a^2) = 0$. At the same time,

$$b \cdot 1 + 1 = b + 1 = a^2 - 1 + 1 = a^2$$
forces $\Delta(a^2) = 1$. Since there will always be a monochromatic solution to either $ax_1 = x_2$ or $bx_1 + x_2 = x_3$ at $a^2$, the disjunctive Rado number must be less than or equal to $a^2$.

Next, it must be shown that there is a good coloring of \{1, \ldots, a^2 - 1\}. Let

$$\Delta(x) = \begin{cases} 0 & \text{if } 1 \leq x \leq a - 1, \\ 1 & \text{if } a \leq x \leq a^2 - 1. \end{cases}$$

This coloring will not have a monochromatic solution to the equation $bx_1 + x_2 = x_3$, as the smallest number that it can produce is

$$b \cdot 1 + 1 = b + 1 = a^2 - 1 + 1 = a^2,$$
which is greater than any number in the coloring. Therefore, the only equation that must be considered is $ax_1 = x_2$. This equation will not produce a monochromatic solution in \{1, \ldots, a - 1\} because

$$a \cdot 1 = a,$$

which is greater than the upper bound of the interval. If 1 is replaced by anything larger in the interval, the resulting product will also be greater than the upper bound of the interval. Likewise, this equation will not produce a monochromatic solution in \{a, \ldots, a^2 - 1\} because

$$a \cdot a = a^2,$$

which is greater than the upper bound of the interval. If $a$ is replaced by anything larger in the interval, the resulting product will also be greater than the upper bound of the interval. Hence, this is a good coloring of \{1, \ldots, a^2 - 1\}, meaning that the disjunctive Rado number in this case is $a^2$. \hfill \Box

**Theorem 6 (Even a Case).** When $b = 2a$ and $a \geq 4$ is even, the disjunctive Rado number is $ab$. 19
Proof. First it must be shown that there will always be a monochromatic solution to either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \) at \( ab \). Let \( \Delta(1) = 0 \). Since

\[
a \cdot 1 = a,
\]

\( \Delta(a) = 1 \). Then,

\[
a \cdot a = a^2 = \frac{b^2}{4},
\]

meaning \( \Delta \left( \frac{b^2}{4} \right) = 0 \). Next,

\[
b \cdot 1 + \frac{b^2}{4} - b = \frac{b^2}{4}
\]

forces \( \Delta \left( \frac{b^2}{4} - b \right) = 1 \). Then,

\[
b \cdot 1 + \frac{b^2}{4} = \frac{b^2}{4} + b,
\]

which means \( \Delta \left( \frac{b^2}{4} + b \right) = 1 \) is necessary. Then,

\[
b \cdot 2 + \frac{b^2}{4} - b = \frac{b^2}{4} + b
\]

forces \( \Delta(2) = 0 \). In turn,

\[
b \cdot 2 + \frac{b^2}{4} = \frac{b^2}{4} + 2b
\]

forces \( \Delta \left( \frac{b^2}{4} + 2b \right) = 1 \). Then, since

\[
b \cdot 3 + \frac{b^2}{4} - b = \frac{b^2}{4} = \frac{b^2}{4} + 3b.
\]

This pattern of forcing \( \Delta \left( \frac{b^2}{4} + kb \right) = 1 \) for increasing integers \( k \geq 1 \) and \( \Delta(k) \) will continue until \( k = \frac{3}{2} - 1 \). Then

\[
b \cdot \frac{a}{2} + \frac{b^2}{4} - b = \frac{b^2}{4} + \left( \frac{a}{2} - 1 \right)b
\]
will force $\Delta \left( \frac{a}{2} \right) = 0$ because $\Delta \left( \frac{b^2}{4} - b \right) = 1$ and $\Delta \left( \frac{b^2}{4} + \left( \frac{a}{2} - 1 \right) b \right) = 1$.

Finally, since $\Delta \left( \frac{a}{2} \right) = 0$ and

$$b \cdot \frac{a}{2} + a^2 = \frac{ab}{2} + \frac{ab}{2} = ab,$$

$\Delta(ab) = 1$. However, since $\Delta(2) = 0$,

$$a \cdot 2 = 2a = b$$

forces $\Delta(b) = 2$, meaning that

$$a \cdot b = ab$$

forces $\Delta(ab) = 0$.

Next it must be shown that there exists a good coloring of $\{1, \ldots, ab - 1\}$ such that there is not a monochromatic solution to either $ax_1 = x_2$ or $bx_1 + x_2 = x_3$. This coloring is

$$\Delta(x) = \begin{cases} 
0 & \text{if } 1 \leq x < a, \\
1 & \text{if } x = a, \\
0 & \text{if } a < x < b, \\
1 & \text{if } b \leq x < a^2, \\
0 & \text{if } x = a^2, \\
1 & \text{if } a^2 < x < ab - 1. 
\end{cases}$$

First the coloring will be analyzed in relation to the equation $ax_1 = x_2$. Since $a \cdot b = ab$ and $ab$ is above the range of the coloring, instances when $x_1 \geq b$ need not be looked at more carefully as they will never result in an equation within the range $\{1, \ldots, ab - 1\}$. Furthermore, since $a \cdot 1 = a$, $a$ is the smallest possible quantity for $x_2$, meaning that any cases where $x_2 < a$ need not be further analyzed. This leaves six cases of possible monochromatic solutions to be examined more carefully.

- The case where $x_1 \in \{1, \ldots, a - 1\}$ and $x_2 \in \{a + 1, \ldots, b - 1\}$ cannot occur because $a \cdot 1 = a$ and $a \cdot 2 = 2a = b$. Since neither $a$ nor $b$ are in $\{a + 1, \ldots, b - 1\}$ and any higher input for $x_1$ would produce an even higher $x_2$, there cannot be a monochromatic solution using $x_1$ and $x_2$ from these respective sets.
• The case where \( x_1 \in \{1, \ldots, a - 1\} \) and \( x_2 = a^2 \) cannot occur because \( a \cdot a - 1 = a^2 - a < a^2 \).

• The case where \( x_1 = a \) and \( x_2 \in \{b, \ldots, a^2 - 1\} \) cannot occur because \( a \cdot a > a^2 - 1 \).

• The case where \( x_1 = a \) and \( x_2 \in \{a^2 + 1, \ldots, ab - 1\} \) cannot occur because \( a \cdot a = a^2 < a^2 + 1 \).

• The case where \( x_1 \in \{a + 1, \ldots, b - 1\} \) and \( x_2 \in \{a + 1, \ldots, b - 1\} \) cannot occur because \( a(a + 1) = a^2 + a \). Since \( a \geq 4 \) and \( b - 1 = 2a - 1 \),
\[
a^2 + a \geq 5a > b - 1.
\]

Therefore, there cannot be a monochromatic solution using \( x_1 \) and \( x_2 \) from this set.

• The case where \( x_1 \in \{a + 1, \ldots, b - 1\} \) and \( x_2 = a^2 \) cannot occur because \( a(a + 1) = a^2 + a > a^2 \).

Since the limited cases that were not earlier eliminated have been shown to be impossible, this coloring will not produce a monochromatic solution to \( ax_1 = x_2 \).

Now that it has been established that this coloring will not produce a monochromatic solution to the equation \( ax_1 = x_2 \), the coloring will be analyzed in relationship to the equation \( bx_1 + x_2 = x_3 \). First, note that
\[
b \cdot a + x_2 > ab,
\]
meaning that instances where \( x_1 \geq a \) need not be analyzed more closely. Furthermore, since
\[
b \cdot 1 + 1 = b + 1,
\]
instances where \( x_3 \leq b \) require no further inspection. The first parameter leaves only the possibility of \( x_1 \in \{1, \ldots, a - 1\} \). The second parameter leaves only the possibility of \( x_3 = a^2 \). Therefore, since
\[
\Delta(1) = \cdots = \Delta(a - 1) = 0,
\]
only instances where \( \Delta(x_2) = \Delta(x_3) = 0 \) need be considered. Since \( x_3 > b \), there are only three cases that need further examination, as \( x_1 \) must always be in \( \{1, \ldots, a - 1\} \) and \( x_3 = a^2 \).

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• The case where \( x_1 \in \{1, \ldots, a - 1\} \), \( x_2 \in \{1, \ldots, a - 1\} \), and \( x_3 = a^2 \) cannot occur because, since \( b = 2a \) and \( x_3 = a^2 \), the equation \( bx_1 + x_2 = x_3 \) can be written as
\[
x_2 = a^2 - 2ax_1 = a(a - 2x_1).
\]
Therefore, since \( a - 2x_1 \) is an integer, \( x_2 \) can either be negative, zero, or \( x_2 \geq a \). Since \( x_2 \) must be positive, \( x_2 > a \), \( x_2 \not\in \{1, \ldots, a - 1\} \). Therefore, there cannot be a monochromatic solution using \( x_1 \), \( x_2 \), and \( x_3 \) as given.

• The case where \( x_1 \in \{1, \ldots, a - 1\} \), \( x_2 \in \{a + 1, \ldots, b - 1\} \), and \( x_3 = a^2 \) cannot occur because, by the same reasoning used in the last case, the equation \( bx_1 + x_2 = x_3 \) can be written as
\[
x_2 = a^2 - 2ax_1 = a(a - 2x_1) = 2a \left(\frac{a}{2} - x_1\right).
\]
Since \( a \) is even, \( \frac{a}{2} \) and thus \( \frac{a}{2} - x_1 \) is an integer. Therefore, \( x_2 \) can either be negative, zero, or \( x_2 \geq 2a \). Since \( 2a = b \) and \( x_2 \) must be positive, \( x_2 > b \), meaning \( x_2 \not\in \{a + 1, \ldots, b - 1\} \). Therefore, there cannot be a monochromatic solution using \( x_1 \), \( x_2 \), and \( x_3 \) as given.

• The case where \( x_1 \in \{1, \ldots, a - 1\} \), \( x_2 = a^2 \), and \( x_3 = a^2 \) cannot occur because \( bx_1 + a^2 > a^2 \).

Since it has now been shown that the three cases that were not previously eliminated cannot produce a monochromatic solution to the equation \( bx_1 + x_2 = x_3 \), it can now be asserted that the given coloring will not produce a monochromatic solution to the equation \( bx_1 + x_2 = x_3 \).

Since it has been shown that there will always be a monochromatic solution to either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \) at \( ab \) and there is a coloring of \( \{1, \ldots, ab - 1\} \) such that there is no monochromatic solution to either \( ax_1 = x_2 \) or \( bx_1 + x_2 = x_3 \), it has been shown that the disjunctive Rado number is \( ab \) when \( b = 2a \) with \( a \) even and \( a \geq 4 \).

\[\square\]

4 Summary

Optimally, it would be best to be able to quantify the disjunctive Rado number for every possible case of \( a \) and \( b \). However, with the complexity of
the interaction between the equations $ax_1 = x_2$ and $bx_1 + x_2 = x_3$, largely stemming from them not being of the same form (e.g., Johnson and Schaal considered equations $ax_1 = x_2$ and $bx_1 = x_2$, where the only variance between the two equations is the coefficient of the first variable), the author could not arrive at an overarching equation to describe all instances. However, the Rado number for every instance in which $a \geq b - 1$ has been determined here, as well as certain other special cases. In many cases in which the Rado number was able to be described, the equation $bx_1 + x_2 = x_3$ dominated, meaning that the single-equation Rado number for this equation was the disjunctive Rado number as well.

Further work to be done on this specific Rado number would be to identify other patterns of disjunctive Rado numbers using these two equations, or, although it does not seem to be possible to the author, describe the form of this disjunctive Rado number for all cases. The ultimate pinnacle of research in disjunctive Rado numbers would be to describe the form of a disjunctive Rado number for the equations $\sum_{i=1}^{m-1} a_i x_i = x_m$ and $\sum_{i=1}^{m-1} b_i x_i = x_m$. Also, there are many different variations on this problem that have either not been looked at or not been completely solved. The field of disjunctive Rado numbers is relatively new, meaning that there is much work left open for inquiry.
5 Appendix: Mathematica Work

In this appendix, the process used in Mathematica to find the disjunctive Rado numbers will be shown so that they can be replicated by the interested reader. Through this process, the cells in Table 1 were completed. In this code, $B$ will denote the set of integers $x$ such that $\Delta(x) = 0$ and $R$ will denote the set of integers $x$ such that $\Delta(x) = 1$. Notes as to what each section of code means will be given along the way. The example used here is the disjunctive Rado number for $4x_1 = x_2$ and $6x_1 + x_2 = x_3$, which is 28. While in practice, trial and error was used to determine the disjunctive Rado number, only the process to show that there will always be a monochromatic solution at 28 and the process to show that there is a good coloring of $\{1, \ldots, 27\}$ will be displayed.

First, it must be shown that there will always be a monochromatic solution to $4x_1 = x_2$ or $6x_1 + x_2 = x_3$ at 28.

```
R = {}; B = {1};
```

This first line is used to begin with the initial assumption that $\Delta(1) = 0$ and there are no numbers that are color 1 yet.

```
{4, 7}
```

Then, this line is used to calculate all of the integers that could be forced to be color 1 using the current set of integers in color 0, or in this case just $\{1\}$. Mathematica computes this as an intersection of sets. First we will look at the Union part of the intersection. The first term, R, takes all of the numbers that have already been determined to be in $R$ and carries them into this set. The other terms take the numbers in $B$ and puts them in for every variable of $4x_1 = x_2$ and $6x_1 + x_2 = x_3$ to get all possible solutions that include either one variable in $B$, in the case of the equation $4x_1 = x_2$, or two variables in $B$, in the case of the equation $6x_1 + x_2 = x_3$. Because this will include non-whole numbers, that set is intersected with Range[28], or the set of whole numbers $\{1, \ldots, 28\}$ to leave $R$ with only the numbers from 1 to 28 that are forced to be color 1. In this case, there are only two, 4 and 7.
This process is then run repeatedly, returning new outputs of $R$ and $B$ until there is an intersection produced.

$$B = \text{Intersection}[\text{Range}[28], \text{Union}[B, \text{R}/7, \text{Flatten}[	ext{Outer}[\text{Plus}, 6\text{R}, 1\text{R}]], \text{Flatten}[	ext{Outer}[\text{Plus}, \text{R}, -6\text{R}]/1, \text{Flatten}[	ext{Outer}[\text{Plus}, -1\text{R}, \text{R}]/6], 4\text{R}, \text{R}/4]]]$$

$$\{1, 16, 28\}$$

$$R = \text{Intersection}[\text{Range}[28], \text{Union}[R, \text{B}/7, \text{Flatten}[	ext{Outer}[\text{Plus}, 6\text{B}, 1\text{B}]], \text{Flatten}[	ext{Outer}[\text{Plus}, \text{B}, -6\text{B}]/1, \text{Flatten}[	ext{Outer}[\text{Plus}, -1\text{B}, \text{B}]/6], 4\text{B}, \text{B}/4]]]$$

$$\{2, 4, 7, 10, 22\}$$

$$B = \text{Intersection}[\text{Range}[28], \text{Union}[B, \text{R}/7, \text{Flatten}[	ext{Outer}[\text{Plus}, 6\text{R}, 1\text{R}]], \text{Flatten}[	ext{Outer}[\text{Plus}, \text{R}, -6\text{R}]/1, \text{Flatten}[	ext{Outer}[\text{Plus}, -1\text{R}, \text{R}]/6], 4\text{R}, \text{R}/4]]]$$

$$\{1, 2, 3, 8, 10, 14, 16, 19, 22, 26, 28\}$$

$$\text{Intersection}[R, B]$$

$$\{2, 10, 22\}$$

This process is designed to produce the numbers that are forced to be a certain color. Therefore, since the set $\{2, 10, 22\}$ is produced when the intersection of $R$ and $B$ is taken, then it is known that there must be a monochromatic solution when 28 is allowed. While this monochromatic solution does not occur at 28, if it can be shown that there is a good coloring of $\{1, \ldots, 27\}$, then the disjunctive Rado number is 28 in this instance.

In order to show that there is a good coloring of $\{1, \ldots, 27\}$, we begin by allowing $\Delta(1) = 0$ and run the process until it produces no additional numbers.

$$R = \{\}; B = \{1\};$$

$$R = \text{Intersection}[\text{Range}[27], \text{Union}[R, \text{B}/7, \text{Flatten}[	ext{Outer}[\text{Plus}, 6\text{B}, 1\text{B}]], \text{Flatten}[	ext{Outer}[\text{Plus}, \text{B}, -6\text{B}]/1, \text{Flatten}[	ext{Outer}[\text{Plus}, -1\text{B}, \text{B}]/6], 4\text{B}, \text{B}/4]]]$$

$$\{4, 7\}$$
{1, 16}

{4, 7, 10, 22}

{1, 2, 3, 16}

{4, 7, 8, 9, 10, 12, 13, 14, 15, 19, 20, 21, 22}

{1, 2, 3, 5, 16}

{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23}

{1, 2, 3, 5, 16}

{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 19, 20, 21, 22, 23}

{1, 2, 3, 5, 16}

Since the last two calculations for B have produced identical sets, we know that there are no more numbers that are forced to be a certain color. Therefore, we must now check to ensure that there is no forced monochro-
matic solution; that is, there is no intersection between the two sets.

\[ \text{Intersection}[R, B] \]
\[
\{\}
\]

Since the intersection of \( R \) and \( B \) is the null set, there are no forced monochromatic solutions of either \( 4x_1 = x_2 \) or \( 6x_1 + x_2 = x_3 \). Next, we must make sure that this is a complete coloring of \( \{1, \ldots, 27\} \). We do this by taking the union of \( R \) and \( B \), which will give us all of the numbers that have already been colored, and taking the complement of that over the set \( \{1, \ldots, 27\} \), indicated below by \( \text{Range}[27] \).

\[ \text{Complement}[\text{Range}[27], \text{Union}[R, B]] \]
\[
\{6, 18, 24, 25, 26, 27\}
\]

However, since there are numbers that remain in this complement, the above sets of \( R \) and \( B \) do not constitute a full coloring of \( \{1, \ldots, 27\} \). Therefore, we must make a full coloring be assigning a color to these numbers. We can select either color, since they are not forced to be a certain color by either set of numbers.

First, we will let 6 be color 0.

\[ B = \text{Union}[B, \{6\}] \]
\[
\{1, 2, 3, 5, 6, 16\}
\]

Then, we must run the calculations for \( R \) and \( B \) again to see if there are any further numbers forced by the assigning of color 0 to 6. Once again, we will stop running the calculation when two consecutive outputs for \( B \) or two consecutive outputs for \( R \) are identical.

\[ R = \text{Intersection}[\text{Range}[27], \text{Union}[R, B/7, \text{Flatten}[\text{Outer}[\text{Plus}, 6*B, 1*B]], \text{Flatten}[\text{Outer}[\text{Plus}, B, -6*B]/1], \text{Flatten}[\text{Outer}[\text{Plus}, -1*B, B]/6], 4B, B/4]] \]
\[
\{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24\}
\]

\[ B = \text{Intersection}[\text{Range}[27], \text{Union}[B, R/7, \text{Flatten}[\text{Outer}[\text{Plus}, 6*R, 1*R]], \text{Flatten}[\text{Outer}[\text{Plus}, R, -6*R]/1], \text{Flatten}[\text{Outer}[\text{Plus}, -1*R, R]/6], 4R, R/4]] \]
\{1, 2, 3, 5, 6, 16\}

Since two consecutive outputs for \( B \) were the same, we stopped the process. Now, we must check to ensure that the assignment of color 0 to 6 and any further forced colorings have not produced a monochromatic solution.

\[
\text{Intersection}[R, B] \\
{} \\
\text{Since the intersection is empty, there has not been a monochromatic solution introduced. Next, we must check to see if this is now a good coloring of \{1, \ldots, 27\}.} \\
\text{Complement}[\text{Range}[27], \text{Union}[R, B]] \\
\{25, 26, 27\} \\
\text{Since there are still numbers remaining in the complement, we must do further work allowing these numbers to be a certain color. We will continue this process by letting the first number remaining in the complement to be color 0 and performing the same process as was done with 6. If no monochromatic solutions arise and nothing remains in the complement, then a good coloring of \{1, \ldots, 27\} will have been established.} \\
B = \text{Union}[B, \{25\}] \\
\{1, 2, 3, 5, 6, 16, 25\} \\
R = \text{Intersection}[\text{Range}[27], \text{Union}[R, B/7, \\
\text{Flatten}[\text{Outer}[\text{Plus}, 6*B, 1*B]], \text{Flatten}[\text{Outer}[\text{Plus}, B, -6*B]/1], \\
\text{Flatten}[\text{Outer}[\text{Plus}, -1*B, B]/6], 4B, B/4]] \\
\{4, 7, 8, 9, 10, 11, 12, 13, 14, 15, 17, 18, 19, 20, 21, 22, 23, 24\} \\
\text{Intersection}[R, B] \\
{} \\
\text{Complement}[\text{Range}[27], \text{Union}[R, B]] \\
\{26, 27\} \\
B = \text{Union}[B, \{26\}]
Since there are no remaining elements in the complement, we now have a good coloring of \{1, \ldots, 27\}. And since we have shown that there will always be a monochromatic solution of $4x_1 = x_2$ or $6x_1 + x_2 = x_3$ when 28 is allowed and there is a good coloring of \{1, \ldots, 27\}, we can now state that 28 is the disjunctive Rado number for these two equations.
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