ENUMERATION OF AUSTRIAN SOLITAIRE

by

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of the requirements
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in Mathematics.

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________________________________________
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Acknowledgement

I would like to utilize this opportunity to extend sincere appreciation towards my thesis supervisor, Dr. Brian Hopkins, without whose guidance, support, and countless revisions, from the initial to the final step of the process, this thesis would not have existed today. I would also like to thank Dr. Rachel Wifall and the Honors Program at Saint Peters College for providing me with the opportunity to write a thesis as an Honors student. I offer my regards to all of those who supported me in any respect during the completion of the thesis.

I dedicate this thesis to my family, especially to my late grandfather Buddhi Sagar Basta-
tola, who passed away last November.
Abstract

This thesis concerns a variant of Bulgarian Solitaire, called Austrian solitaire, introduced by Akin and Davis. A primary result is the derivation of a formula for the number of states under Austrian Solitaire. This thesis characterizes the Garden of Eden states. The thesis also gives a possible characterization for the fixed points and examines other cycle states with various conclusions.
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1 Integer Partitions and Generating Functions

Partitions of a positive integer are the ways the number can be written as a sum of other positive integers without regard to order. For example,

\[
\begin{align*}
4 &= 4 \\
&= 3 + 1 \\
&= 2 + 2 \\
&= 2 + 1 + 1 \\
&= 1 + 1 + 1 + 1
\end{align*}
\]

The partitions of 4 are (4), (3,1), (2,2), (2,1,1), and (1,1,1,1). The length of a partition is the number of integers, called parts, in the partition. Length of the partition (2,1,1) is 3.

Using generating function is a way of counting the number of integer partitions. Let \( p(n) \) denote the number of integer partitions of \( n \). Let \( p(n,k) \) be the number of partitions of \( n \) with largest part at most \( k \) and \( p(n,k,m) \) be the number of partitions of \( n \) with largest part at most \( k \) and length less than or equal to \( m \).

The generating function for an unrestricted partitions \( p(n) \) is

\[
\prod_{i=1}^{\infty} \frac{1}{1-x^i} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots
\]

since each factor \( \frac{1}{1-x^i} = 1 + x^i + x^{2i} + \cdots \) accounts for \( i \) occurring in partitions. The coefficient of \( x^n \) denotes the number of partition for an integer \( n \). From above we can conclude that
there are 5 partitions of 4, i.e., $p(4) = 5$.

The generating function for $p(n, k)$, partitions of $n$ with largest part at most $k$, is

$$\prod_{i=1}^{k} \frac{1}{1 - x^i} = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots) \cdots (1 + x^k + x^{2k} + \cdots)$$

For example, the generating function to count the partitions of $n$ with the largest part at most 3 is

$$\prod_{i=1}^{3} \frac{1}{1 - x^i} = 1 + x + 2x^2 + 3x^3 + 4x^4 + \cdots$$

Here, we can see that there are only 4 partitions of 4 with largest part at most 3. This is because the partition (4) is not counted. Hence, $p(4, 3) = 4$.

We can also count the partitions of different length using generating functions. “Euler’s trick” of multiplying $z$ with $x^i$ keeps the track of the number of parts in a partition. This allows us to count the partitions of $n$ with and maximum length $m$. This generating function is

$$\prod_{i=1}^{\infty} \frac{1}{1 - z x^i} = 1 + z x + (z + z^2) x^2 + (z + z^2 + z^3) x^3 + (z + 2z^2 + z^3 + z^4) x^4 + \cdots$$

Here, the power of $x$ denotes what integer partition $n$ we are looking at. The power of $z$ denotes the length of the partition and the coefficient of $z$ gives us how many partition there are for that particular length and particular $n$. In the equation above, for $x^4$, i.e., $n = 4$, the
powers and coefficients of $z$ denotes that there is 1 partition of length 1, 2 of length 2, 1 of length 3, and 1 of length 4.

To combine restrictions on largest part and number of parts, use the generating function $\prod_{i=1}^{k} 1/(1 - zx^i)$. The value $p(n, m, k)$, the number of partitions of $n$ with largest part $k$ and maximum length $m$, is the sum of the coefficients of $z^m x^n, z^{m-1} x^n, \ldots, z x^n$. For example $p(4, 3, 3) = 3$ from $z^2 x^{3+1}, z^2 x^{2+2}, z^3 x^{2+1+1}$. Notice that $zx^4$ does not arise because of the restriction on largest part, and $z^4 x^{1+1+1+1}$ is excluded because of the restriction on partition length.

2 Bulgarian Solitaire

Bulgarian solitaire is an example of dynamics on partitions of a positive integer. Bulgarian Solitaire can be seen in terms of a random card game. In the game, a pack of $n$ cards can be divided into several piles. For each pile one card is removed and the removed cards are collected together to form a new pile (piles of size zero are ignored). The game is continued until a fixed point or a cycle is reached. It has been proven that when $n$ is equal to $k^{th}$ triangular number, $T_k = k(k + 1)/2$, no matter which partition of $n$ we start the game with, we always end up at the partition $(k, k - 1, \ldots, 2, 1)$. If $n$ is not a triangular number we always end up in a cycle of length 2 or more. For some $n$ there are multiple cycles. [3]

Let $\lambda = (\lambda_1, \ldots, \lambda_t)$ be a partition of $n$, where $\lambda_1, \ldots, \lambda_t$ are $t$ positive integers in non-increasing order whose sum is $n$. The Bulgarian Solitaire operation, $B$, on $\lambda$ is defined as $B(\lambda) = (t, \lambda_1 - 1, \ldots, \lambda_t - 1)$ ignoring any zeros and reordering in an non-increasing order if
Ferrers diagrams provide graphical representations of partitions. We use $\lambda_1$ dots in the first column of the Ferrers diagram followed by a column of $\lambda_2$ dots, etc. Figure 1 is an example of both Ferrers diagrams and the Bulgarian Solitaire operation.

![Ferrers Diagram Example](image)

Figure 1: $B(4, 1, 1, 1, 1) = (5, 3)$.

Bulgarian Solitaire has been researched extensively. The Garden of Eden partitions, that is, the partitions with no pre-image, have been characterized and counted for a given $n$. The cycle partitions have also been characterized and all cycle lengths are known. [1, 2, 3]

In Figure 2, we can see that for $n = 8$, Bulgarian Solitaire does not have a unique cycle, rather there are two cycles, a 4-cycle and a 2-cycle.
3 Introduction to Austrian Solitaire

Austrian Solitaire is a variant of Bulgarian Solitaire, where parts are restricted and a reserve is added.

Austrian Solitaire starts by laying out deck of \( n \) cards with all stacks of size less than \( L \), where \( L \) is a fixed number less than or equal to \( n \). There is one separate special stack which is called the bank. A round of play has two steps. In the first step one card is picked from each of the stacks, except the bank, and put in the bank. In the next step, we lay out stacks of size \( L \) from the bank until it has fewer than \( L \) cards. Let \( b \) denote the number of cards left in the bank; note that \( 0 \leq b \leq L - 1 \).

Akin and Davis attributed the name of the game to the Austrian school of capital theory,
Think of the ordinary stacks as machines. Each machine has, when new, a life of exactly $L$ years. The size of a stack is the number of productive years left for a particular machine. Each year it ages one year (and so one card is removed from the stack). For each machine on line the company deposits $1/L$ of its cost into the bank as a sinking fund. Then it buys as many new machines as it can afford, and the remaining funds are left in the bank until next year. [1]

Akin and Davis conjectured that, for any fixed $n$ and $L$, there is a unique cycle, unlike Bulgarian Solitaire, where there could be more than one cycles.

A state in Austrian solitaire consists of a partition of $n - b$ and a bank $b$ where $0 \leq b \leq L - 1$. Let $(\lambda|b)$ denote a state with partition $(\lambda_1, \ldots, \lambda_t)$ and bank $b$, where $t$ is the number of parts in the partition. Sometimes we will write $(\lambda|b)$ as $(L^k, \lambda_{k+1}, \ldots, \lambda_t|b)$, where $k$ is the number of $L$’s in the partition.

Define $\lfloor x \rfloor$ to be the nearest integer less than or equal to $x$ (the “floor”) and $\lceil x \rceil$ as the nearest integer greater than or equal to $x$ (the “ceiling”). The Austrian Solitaire operation is given by

$$A(\lambda|b) = (L^{\lfloor \frac{t+b}{L} \rfloor}, \lambda_1 - 1, \ldots, \lambda_t - 1 | (b + t) \mod L)$$

ignoring any zeros that might occur in the partitions. Taking one from each pile gives $t$ in the bank, in addition to $b$ that was already there. There are $\lfloor \frac{t+b}{L} \rfloor$ new $L$ parts generated,
and \((b+t) \mod L\) left in the bank. Notice that, unlike Bulgarian Solitaire, reordering is never required because only the largest possible parts can be added. Figure 3 shows an example of the Austrian Solitaire operation.

Figure 3: \(A(2|4) = (5, 1|0)\) for \(n = 6\) and \(L = 5\).

Figure 4 and Figure 5 are two examples of all states under Australian Solitaire for certain parameters. Figure 4 shows Austrian Solitaire when \(n = 6\) and \(L = 2\). All states lead to a fixed point, a 1-cycle. In Figure 5, where \(n = 6\) and \(L = 5\), all states lead to a 4-cycle.

Figure 4: Australian solitaire on \(n = 6\) and \(L = 2\).

Figure 5: Austrian Solitaire on \(n = 6\) and \(L = 5\)
4 Number of States

Unlike Bulgarian Solitaire, which has $p(n)$ states, it is not obvious how many states Austrian Solitaire has for a fixed $n$ and $L$ since there are states with positive bank and partitions smaller than $n$.

Let $A(n, L)$ be the total number of states for a given $n$ and bank parameter $L$. Table 1 gives the total number of states for small values of $n$ and $L$. In this section we derive a formula for $A(n, L)$ in terms of the known partition functions $p(n, k)$ and $p(n, k, m)$ described in Section 1.

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Table 1: $A(n, L)$

We know that any state with a partition of $n$ with the largest part at most $L$ is in our system, and we are able to count these states easily. These are all the states with zero bank, $(\lambda|0)$. Table 2 gives $p(n, L)$, the number of partitions of $n$ with largest part at most $L$.

It remains to count the states $(\lambda|b)$ where $b$ is positive. Table 3 gives number of states
Table 2: $p(n, L)$

with positive bank for small values of $n$ and $L$.

Table 3: Number of states with positive bank.

For any given $n$ and $L$, not every pair $(\lambda|b)$ with $1 \leq b \leq L - 1$ and $\lambda$ a partition of $n - b$ is a valid state. For example, with $n = 9$ and $L = 4$, we see that $(2,1,1,1,1,1,1|1)$ is not a valid state. The partition of this state would have to come from $(3,2,2,2,2,2)$, a partition of 15, a contradiction. This leads to the following characterization of states with
Lemma 1. For \(1 \leq b \leq L - 1\) and \(\lambda = (L^k, \lambda_{k+1}, \ldots, \lambda_t)\) a partition of \(n - b\), the pair \((\lambda|b)\) is a valid state if and only if \(\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n\).

Proof. For any state with \(\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n\), we can always find a preimage state with \(b = 0\), which is \((\lambda_{k+1} + 1, \ldots, \lambda_t + 1, 1^{kL-(t-k)+b}|0)\). We know that any state with \(b = 0\) and largest part less than or equal to \(L\) is a valid state. So any state with \(\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n\) is part of Austrian Solitaire.

For \(1 \leq b \leq L - 1\), if the sum \(\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 > n\), then \((\lambda|b)\) is not a valid state because the pre-image would have a partition of a number greater than \(n\), which is not possible. Hence, for any valid state with positive bank \(\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n\). \(\square\)

The proof of Lemma 1 also proves the following result.

Corollary 1. A state with positive bank is not more than 1 step from a some state with bank zero.

We can now derive a formula for \(A(n, L)\).

Theorem 1. \[
A(n, L) = p(n, L) + \sum_{i=1}^{L-1} \sum_{j=\lfloor \frac{n}{L} \rfloor}^{\lfloor \frac{n-i}{L} \rfloor} p(n - i - Lj, L - 1, i + Lj)
\]

Proof. The total number of states with zero bank is given by \(p(n, L)\). The rest of the formula counts the states with positive bank.
Let $i$ denote the parameter for the bank size which is greater than or equal to 1 and less than or equal to $L - 1$ for a positive bank state. From Lemma 1, we know that a positive bank state is an image of a zero bank state. The length of a zero bank state has to be at least $\lceil n/L \rceil$ for a given $n$ and $L$. Therefore, positive bank state has to have at least $\lceil n/L \rceil / L$ parts $L$. Let $j$ denote the parameter for the number of $L$’s in the partition of the positive state; $j$ cannot exceed $[(n - i)/L]$. Hence, $\lceil n/L \rceil / L \leq j \leq [(n - i)/L]$.

In order to count the states with positive bank, we count their partitions after any $L$’s are removed.

The number of partitions of $n$ with largest part at most $k$ and maximum length $m$, described in Section 1, is $p(n, k, m)$. Since we are excluding $j$ copies of $L$, we are counting partitions of the integer $n-i-Lj$ where the largest possible part is $L-1$. Again from Lemma 1 we can conclude that the length of the partition cannot exceed the difference between $n$ and the partition sum. Hence the maximum length allowed is $n - (n - i - Lj) = i + Lj$.

Hence, the total number of states with positive bank is given by

$$
\sum_{i=1}^{L-1} \sum_{j=\left\lfloor \frac{n-i}{L} \right\rfloor}^{\left\lceil \frac{n}{L} \right\rceil / L} p(n-i-Lj, L-1, i+Lj)
$$

Therefore, total number of states under the system is

$$
A(n, L) = p(n, L) + \sum_{i=1}^{L-1} \sum_{j=\left\lfloor \frac{n-i}{L} \right\rfloor}^{\left\lceil \frac{n}{L} \right\rceil / L} p(n-i-Lj, L-1, i+Lj)
$$

\[\square\]
As an example of applying the Theorem 1, consider the case for \( n = 9 \) and \( L = 4 \). Notice that the lower bound for \( j \) is 0 since \( \lfloor \lceil 9/4 \rceil / 4 \rfloor = \lfloor 3/4 \rfloor \).

\[
A(9, 4) = p(9, 4) + \sum_{i=1}^{3} \sum_{j=0}^{\lfloor \frac{9-i}{4} \rfloor} p(9 - i - 4j, 3, i + 4j)
\]

\[
= p(9, 4) + p(9 - 1 - 0, 3, 1) + p(9 - 1 - 4, 3, 5) + p(9 - 1 - 8, 3, 9)
\]

\[
+ p(9 - 2 - 0, 3, 2) + p(9 - 2 - 4, 3, 6) + p(9 - 3 - 0, 3, 3) + p(9 - 3 - 4, 3, 7)
\]

\[
= p(9, 4) + p(8, 3, 1) + p(4, 3, 5) + p(0, 3, 9)
\]

\[
+ p(7, 3, 2) + p(3, 3, 6) + p(6, 3, 3) + p(2, 3, 7)
\]

\[
= p(9, 4) + p(8, 3, 1) + p(4, 3) + p(0) + p(7, 3, 2) + p(3) + p(6, 3, 3) + p(2)
\]

\[
= 18 + 0 + 4 + 1 + 0 + 3 + 3 + 2
\]

\[
= 31
\]

matching the entry in Table 1 which was found by directly analyzing the system (see the appendix for the Mathematica code used to generate data).

5 Garden of Eden States

Garden of Eden (GE) states are the states that do not have a preimage. In Figure 5, \((2, 1, 1, 1, 1 | 0)\) is a GE state since there is no state that leads to it. But certainly not every zero bank state is Garden of Eden. \((5, 1 | 0)\) is not a GE state because it is an image of \((2, 1, 1, 1, 1 | 0)\). We can now characterize all such states for a given \( n \) and \( L \).
Theorem 2. The GE states for given $n$ and $L$ are of the form $(L^k, \lambda_{k+1}, \ldots, \lambda_t|0)$ where $\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 > n$.

Proof. Since Lemma 1 shows that any state with a positive bank has a preimage, all GE states must be of the form $(L^k, \lambda_{k+1}, \ldots, \lambda_t|0)$. For states with $\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n$ we can find a preimage state, $(\lambda_{k+1} + 1, \ldots, \lambda_t + 1, 1^{kL-(t-k)}|0)$. The states with $\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 > n$ would have to come from a state with deck size higher than $n$, which is not possible. 

Table 4 gives the number of GE states for small $n$ and $L$. Notice that it is identical to Table 3.

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Table 4: Number of GE states.

Conjecture 1. There is a bijection between the Garden of Eden states and the states with positive bank.

The following lemma may be helpful for establishing a bijection that would prove the conjecture.
Lemma 2. If $(\lambda|0)$ is not a GE state then $L$ is a part in the partition of the state.

Proof. From Theorem 1, we can conclude that $(\lambda|0)$ is not GE if $\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq n$.

Since $b = 0$, we know $kL + \lambda_{k+1} + \cdots + \lambda_t = n$.

Combining these two facts gives

$$\lambda_{k+1} + 1 + \cdots + \lambda_t + 1 \leq kL + \lambda_{k+1} + \cdots + \lambda_t$$

$$t - (k + 1) + 1 \leq kL$$

$$t \leq kL + k$$

When $k = 0$, this gives $t \leq 0$, which is impossible since any partition must have at least one part. Hence, $k \geq 1$, i.e., at least one part of partition of $(\lambda|0)$ must be $L$, for the state to not be a GE partition.

\hfill \Box

6 Cyclic States

Let $C(n, L)$ be the number of cyclic states for a given $n$ and $L$. Table 5 gives number of cyclic states for small values of $n$ and $L$. We can understand the case when $C(n, L) = 1$ and prove various results for other cases that may help lead to a resolution of the Akin-Davis conjecture. As in Bulgarian Solitaire, the triangular numbers play an important role.
Table 5: $C(n, L)$

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**Theorem 3.** $C(n, L) = 1$ for all $n = k \cdot T_L + b$ with $0 \leq b \leq L - 1$ and any positive integer $k$. In particular, the fixed point is $(L^k, (L - 1)^k, \ldots, 2^k, 1^k | b)$.

**Proof.** Consider Austrian Solitaire operation in two steps: in the first step extract $kL$, the length of the partition, from the partition and bring it to the bank. The state becomes $((L - 1)^k, (L - 2)^k, \ldots, 1^k | b + kL)$. Now bring forward $k$ parts $L$ from the bank. The image for the given state is $(L^k, (L - 1)^k, \ldots, 2^k, 1^k | b)$, which is the state itself.

The partition of a state can be of form $(L^k, (L - 1)^k, \ldots, 2^k, 1^k)$ only for $n = k \cdot T_L + b$.  

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In Table 5, Theorem 3 is demonstrated in the 1’s appearing in columns, such as $n = 10, 11, 12, 13$ for $L = 4$ and $n = 6, 7, 8$ for $L = 3$. We believe that the condition in Theorem 3 is a necessary and sufficient condition for all fixed points.

Theorem 3 addresses the 1’s in Table 5. The following lemma addresses the repeating patterns of larger integers in the table. Denote by $(\kappa|0) \cup (\lambda|b)$ the state with bank $b$ whose partition is the union of the parts in the partitions $\kappa$ and $\lambda$. Write $(L, L - 1, \ldots, 1|0)$ as $(\Delta_L|0)$.

Lemma 3. Given a cyclic state $(\lambda|b) = (\lambda_1, \ldots, \lambda_t|b)$ for $n$ and $L$, the state $(\Delta_L|0) \cup (\lambda|b)$ is also cyclic.

Proof. Since $(\lambda|b)$ is a cyclic state, $A(\lambda|b) = (L^\lfloor\frac{t+b}{L}\rfloor, \lambda_1 - 1, \ldots, \lambda_t - 1|(b + t) \mod L)$ is also a cyclic state. The union $(\Delta_L|0) \cup (\lambda|b) = (L, L - 1, \ldots, 1, \lambda_1, \ldots, \lambda_t|b)$

(with parts reordered as necessary) has image $A((\Delta_L|0) \cup (\lambda|b)) = (L, L - 1, \ldots, 1, L^\lfloor\frac{t+b}{L}\rfloor, \lambda_1 - 1, \ldots, \lambda_t - 1|(b + t) \mod L)$

That is, $A((\Delta_L|0) \cup (\lambda|b)) = (\Delta_L|0) \cup A(\lambda|b)$. The parts in $\Delta_L$ can be simply combined with partitions of all the cyclic states for a given $n$ and $L$ to produce the cyclic states for $n + T_L$ and $L$. Hence, $(\Delta_L|0) \cup (\lambda|b)$ is a cyclic state. \hfill \Box

Again, we believe that this also a necessary and sufficient condition. If so, the first
$T_L - 1$ values in column $L$ of Table 5 and the following $L$ occurrences of 1 all repeat for all $n \geq T_L + L - 1$.

Notice that the union of the partitions of the four cycle states in Figure 5 is $\Delta_5$. This is an example of the following result.

**Lemma 4.** Each of $L, L - 1, \ldots, 2, 1$ appear same number of times in the partitions of a cycle for a given $n$ and $L$.

*Proof.* Some cycle state must include $L$ as a part, else it would be impossible to cycle back to the same state since the Austrian Solitaire reduces the size of each part. Each $L$ in a cycle produces an $L - 1$, an $L - 2$, etc., down to 1. All partition parts in cycle states appear this way. Hence, if there are $k$ parts $L$ in the union of the partitions of the cycle states, then there are also $k$ parts $L - 1$, etc. \qed

We end with two related conjectures supported by experimental data.

**Conjecture 2.** Each possible bank size appears at most once among the cycle states for given $n$ and $L$.

The following conjecture would follow, but may have an independent proof.

**Conjecture 3.** For given $n$ and $L$, the number of cyclic states is less than or equal to $L$.
7 Conclusion

This thesis studies the relatively unexplored world of Austrian Solitaire. The thesis counts the number of states occurring under Austrian Solitaire. It characterizes the Garden of Eden states and provides a conjecture regarding these states, which if proven will count the Garden of Eden states. It also characterizes the fixed points and proves some results that could help in proving the conjecture made by Akin and Davis in 1985. This thesis began with the objective to prove the conjecture. Even though it was not able to do so, I believe that it helps understand Austrian Solitaire better and paves a path to prove the final conjecture.
Appendix

1. Austrian Solitaire using Mathematica.

\[
<< \text{Combinatorica}
\]

\[
A[p_, b_, L_] := \text{DeleteCases} \left[ \text{Flatten} \left[ \text{Table}[L, \{i, \text{Quotient}[\text{Length}[p] + b, L]\}, p - 1]\right], 0\right], \text{Mod}[\text{Mod}[\text{Length}[p], L] + b, L], L
\]

\(A(p, b, L)\) is the Mathematica operation for Austrian Solitaire on a state with partition \(p\), current bank size \(b\) and the bank parameter \(L\). Here are some examples of the operation.

\[
A[\{3,1,1,1\},0,5]
\]

\[
\{2\},4,5
\]

\[
A[\{2\},4,5]
\]

\[
\{5,1\},0,5
\]

The following commands determine all the states for \(n = 6\) and \(L = 5\).

\[
\text{ZeroethLevel}_{65} = \text{Table}[\{\text{IntegerPartitions}[6, \text{All}, \text{Range}[5]][[i]], 0, 5\}, \{i, \text{Length}[\text{IntegerPartitions}[6, \text{All}, \text{Range}[5]]]\}]
\]

\[
\{\{5,1\},0,5\}, \{\{4,2\},0,5\}, \{\{4,1,1\},0,5\}, \{\{3,3\},0,5\}, \{\{3,2,1\},0,5\}, \{\{3,1,1,1\},0,5\}, \{\{2,2,2\},0,5\}, \{\{2,2,1,1\},0,5\}, \{\{2,1,1,1,1\},0,5\}, \{\{1,1,1,1,1,1\},0,5\}
\]

\[
\text{FirstLevel}_{65} = \text{Table}[A[\text{ZeroethLevel}_{65}[[i]], \{i, \text{Length}[\text{ZeroethLevel}_{65}]\}]]
\]

\[
\{\{4\},2,5\}, \{\{3,1\},2,5\}, \{\{3\},3,5\}, \{\{2,2\},2,5\}, \{\{2,1\},3,5\}, \{\{2\},4,5\}, \{\{1,1\},3,5\}, \{\{1,1\},4,5\}, \{\{5,1\},0,5\}, \{\{5\},1,5\}
\]
\[ \text{Set}_{65} = \text{Union[ZeroethLevel}_{65}, \text{FirstLevel}_{65}] \]

\[ \{\{2\}, 4, 5\}, \{\{3\}, 3, 5\}, \{\{4\}, 2, 5\}, \{\{5\}, 1, 5\}, \{\{1, 1\}, 4, 5\}, \{\{2, 1\}, 3, 5\}, \{\{2, 2\}, 2, 5\}, \{\{3, 1\}, 2, 5\}, \{\{3\}, 0, 5\}, \{\{4, 2\}, 0, 5\}, \{\{5, 1\}, 0, 5\}, \{\{1, 1, 1\}, 3, 5\}, \{\{2, 2, 2\}, 0, 5\}, \{\{3, 2, 1\}, 0, 5\}, \{\{4, 1, 1\}, 0, 5\}, \{\{2, 2, 1, 1\}, 0, 5\}, \{\{3, 1, 1, 1\}, 0, 5\}, \{\{2, 1, 1, 1, 1\}, 0, 5\}, \{\{1, 1, 1, 1, 1\}, 0, 5\} \]

ZeroethLevel_{nL} gives us all the zero bank states. The FirstLevel_{nL} command gives the states that are the images of the states of ZeroethLevel_{nL}. We know that we do not have to go further than FirstLevel_{nL} by Corollary 1. Then Set_{nL} gives the union of the ZeroethLevel_{nL} and FirstLevel_{nL} to give all the states for a given \( n \) and \( L \).

\[ \text{Length[Set}_{65}] \]

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\[ \text{G}_{65} = \text{FunctionalGraph[A, Set}_{65}] \]

-Graph: <19, 19, Directed> —

\[ \text{GraphPlot[G}_{65}, \text{SelfLoopStyle} \rightarrow \text{True, MultiedgeStyle} \rightarrow \text{True, VertexLabeling} \rightarrow \text{True}] \]
\textbf{Set65}[[\{1, 2, 3, 11\}]]

\{\{2\}, 4, 5\}, \{\{3\}, 3, 5\}, \{\{4\}, 2, 5\}, \{\{5\}, 1, 0, 5\}\}

The figure is the graph of all the states under Austrian Solitaire for $n = 6$ and $L = 5$. This corresponds to Figure 5. The vertices labelled 1, 2, 3 and 11 are the cyclic states.
References


